Banach Algebra Amenability and Group Algebras

ANDREW GORDON KEPERT

May, 1992

A thesis submitted for the degree of Doctor of Philosophy
of The Australian National University

Declaration

This thesis consists of original work carried out by myself, under the supervision of Dr Rick Loy and Dr George Willis. Any contributions by my supervisors and other colleagues that goes beyond their providing a fertile environment have been indicated as such in the text of this thesis.

Some parts of the research presented in this thesis—primarily that in Chapter 1—have appeared previously in an internal publication of the School of Mathematical Sciences at the Australian National University. This was Mathematics Research Report 022-91, entitled "The range of group algebra homomorphisms", of which I was the sole author. This paper has subsequently been submitted for publication with the Journal of Functional Analysis.

15/5/92

Revised 24/1/93

Acknowledgement

I would like to thank my supervisors at the Australian National University, Dr R.J. Loy and Dr G.A. Willis for their constant encouragement and many helpful suggestions, and more recently, their patience in proof-reading drafts of this thesis. Many thanks also to the other mathematicians whom I have consulted, or who have been consulted on my behalf, in particular Professors B.E. Johnson, P.C. Curtis, Jr. and U. Haagerup for their suggestions on specific problems that found their way to being brought to my attention, and to Dr N. Grønbæk, for the suggestion that spawned Chapter 3. Special thanks are due to my wife, Sharyn Fawcett, to whom I dedicate this thesis, for remaining supportive throughout, and ensuring that I retain my humanity.

Abstract

The primary purpose of the research presented in this thesis is to investigate the extent to which amenability in Banach algebras is reliant on amenability in locally compact groups. One of the first results proven on amenability in Banach algebras was the equivalence of the amenability of a locally compact group G with the amenability of its group algebra $L^1(G)$. Other instances of amenability in Banach algebras can then be demonstrated by the use of certain constructions which preserve amenability.

It is conceivable that we might be able to obtain a characterization of the sort "A Banach algebra $\mathfrak{A}$ is amenable if and only if there is a construction that starts with amenable group algebras, and proceeds at each step via a construction that preserves amenability to finally arrive at $\mathfrak{A}$." Several such constructions are attempted in the first three chapters of the thesis, where it is shown that none of them achieve the desired characterization of amenability.

In the course of this analysis, an incidental result is obtained on the range of a homomorphism between two group algebras. This is a complete characterization that implies, amongst other things, that the range of a homomorphism between commutative group algebras is closed. The fourth chapter deals with possible generalizations of this result—concentrating on conditions on Banach algebras $\mathfrak{A}$ and $\mathfrak{B}$ which ensure that an algebra homomorphism $\nu : \mathfrak{A} \to \mathfrak{B}$ whose range is dense in $\mathfrak{B}$ is onto. Several results are obtained, displaying situations in which a dense-ranged homomorphism is necessarily onto, and other situations where this is not the case. A significant conjecture made here is that if ν is a dense-ranged homomorphism from an amenable Banach algebra into a commutative group algebra, then ν is onto.

The appendices deal with problems further removed from classifying amenability. Appendix A deals with the problem of finding a characterization of the range of a homomorphism between the measure algebras of two locally compact Abelian groups. Some positive results are obtained, similar to the characterization of the range of a homomorphism between commutative group algebras, and there are indications that there may be a procedure which applies generally. This problem is conjectured to be related to the problem to which a small contribution is made in Appendix B—that of finding a Banach space complement to an ideal in a group algebra.

Contents

Chapter 0. Introduction

The development of the theory of Banach algebras has been greatly influenced by ideas from harmonic analysis. This is particularly the case for amenability in Banach algebras—the term itself originated from the property of amenability in locally compact groups, due to the equivalence of the amenability of a locally compact group G and the amenability of its group algebra $L^1(G)$. It has been this relationship between amenability in locally compact groups and amenability in Banach algebras that has provided many of the known examples of amenable Banach algebras—the amenability of these algebras can be related to that of certain amenable group algebras via constructions that preserve amenability. In the first three chapters, we will consider several such constructions, and determine the extent to which they can be used to characterize amenability. In the remainder of this thesis, Chapter 4 and the appendices, we investigate generalizations of some incidental results from Chapter 1.

0.1. Definitions, Notation and Conventions

The notational conventions and basic definitions used in this thesis are summarized below.

The notation for sets and functions is standard, with the following possible exceptions. If A and B are sets, we will write $A \subseteq B$ to signify that A is a subset of B, reserving the notation $A \subset B$ for when $A \neq B$. The relative complement of B in A will be denoted $A \setminus B$, whilst the *symmetric difference* of A and B is defined to be $A \triangle B = (A \setminus B) \cup (B \setminus A)$. If $\{A_i\}_{i \in \mathbb{I}}$ is a family of sets, then the *disjoint union* of these sets is defined to be $\bigcup_{i \in \mathbb{I}} \{i\} \times A_i \subseteq \mathbb{I} \times \left(\bigcup_{i \in \mathbb{I}} A_i\right)$, and will be denoted $\dot{\bigcup}_{i \in \mathbb{I}} A_i$. We will usually consider each A_i to be a subset of $\dot{\bigcup}_{i \in \mathbb{I}} A_i$. The *cartesian product* of $\{A_i\}$ will be denoted $\prod_{i \in \mathbb{I}} A_i$. If $f : A \to B$ and $g : A \to C$ are functions with $g(a_1) = g(a_2) \implies f(a_1) = f(a_2)$, then $f \circ g^{-1}$ will denote the function $g(A) \to B$ given by $f \circ g^{-1}(g(a)) = f(a)$. If $A \subseteq B$, $\chi_A : B \to \mathbb{C}$ will denote the *characteristic function* of B, given by $\chi_A(x) = 1$, if $x \in B$, and $\chi_A(x) = 0$, if $x \in A \setminus B$. If $A \subseteq B$ and $f : B \to C$ is a function, the *restriction* of f to A is denoted $f|_A : A \to C$. This *restriction mapping* $f \mapsto f|_A$ will be denoted $\rho_{B,A}$, or ρ_A. An *extension* of a function $g : A \to C$ is a function $g' : B \to C$ such that $g'|_A = g$. We denote the cardinality of a set X by $|X|$.

Topological concepts and notation are also mostly standard. All topological spaces considered are assumed to be Hausdorff. By a *neighbourhood* of a point x in a topological space, we mean a set $V \subseteq X$ such that there is an open set U with $x \in U \subseteq V$. A *clopen* set is one that is closed and open. If $\{X_i\}_{i \in \mathrm{I}}$ are topological spaces, their *disjoint union* is $X = \dot{\bigcup}_{i \in \mathrm{I}} X_i$ with the unique topology such that each injection $X_i \hookrightarrow X$ is a homeomorphism onto a clopen set. If X and Y are locally compact spaces, a map $f : X \to Y$ is called *proper* if $f^{-1}(C)$ is a compact subset of X, whenever C is a compact subset of Y. (cf. [7, 1.10].) Note that while n is often used to index a net, this does not imply that the net is a sequence.

The groups, linear spaces and algebras considered herein, generally will have a topology (although we may consider the discrete topology on a group) and so by a *homomorphism*, we will mean a *continuous* homomorphism. Similarly, epimorphisms, monomorphisms, isomorphisms and automorphisms will be assumed to be continuous. For Banach spaces and Banach algebras, the last two of these automatically have continuous inverse, by Banach's Isomorphism Theorem. For groups, a *topological* isomorphism is one with continuous inverse, and an automorphism is assumed to be a topological isomorphism. The relation $\cong$ used between groups, normed linear spaces or normed algebras will indicate the existence of a bicontinuous isomorphism. If we want to consider morphisms that are not necessarily continuous, we refer to *algebraic* morphisms. Also, when we speak of a *quotient* of a topological group (respectively linear space, algebra), we mean a quotient of that group by a closed normal subgroup (respectively closed linear subspace, closed two-sided ideal.) Quotient mappings will always be denoted Q_X, where X is the subgroup, subspace, or ideal by which the quotient is taken.

Banach space notation and theory will also be standard. All Banach spaces will be vector spaces over the field $\mathbb{C}$. We will use $\mathfrak{X}^*$ for the dual of a Banach space $\mathfrak{X}$, and consider $\mathfrak{X} \subseteq \mathfrak{X}^{**}$ in the natural way. If $T : \mathfrak{X} \to \mathfrak{Y}$ is a continuous linear mapping, then we denote the adjoint of T by $T^* : \mathfrak{Y}^* \to \mathfrak{X}^*$. If $x \in \mathfrak{X}$, $f \in \mathfrak{X}^*$, we may use $\langle x, f\rangle$ or $\langle f, x\rangle$ for $f(x)$. If $S \subseteq \mathfrak{X}$, we use $S^\perp$ for $\{ f \in \mathfrak{X} : \langle x, f\rangle = 0 \ (x \in S)\}$, and if $S \subseteq \mathfrak{X}^*$, we use ${}^\perp S$ for $\mathfrak{X} \cap S^\perp$. If $\{\mathfrak{X}_i\}_{i \in \mathbb{I}}$ is a family of Banach spaces, we define the c_0-*direct sum* and the ℓ^p *direct sum* (for $1 \leq p < \infty$) of these spaces by

$$\bigoplus_{i\in\mathbb{I}}{}_0\ \mathfrak{X}_i = \left\{ \{x_i\}_{i\in\mathbb{I}} \in \prod_{i\in\mathbb{I}} \mathfrak{X}_i : \left\{ \|x_i\|_{\mathfrak{X}_i} \right\}_{i\in\mathbb{I}} \in c_0(\mathbb{I}) \right\},$$

$$\bigoplus_{i\in\mathbb{I}}{}_p\ \mathfrak{X}_i = \left\{ \{x_i\}_{i\in\mathbb{I}} \in \prod_{i\in\mathbb{I}} \mathfrak{X}_i : \left\{ \|x_i\|_{\mathfrak{X}_i} \right\}_{i\in\mathbb{I}} \in \ell^p(\mathbb{I}) \right\}$$

respectively. With norms $\left\|\{x_i\}\right\|_0 = \sup\|x_i\|$ and $\left\|\{x_i\}\right\|_p = \left(\sum\|x_i\|^p\right)^{1/p}$ respectively, these are Banach spaces. We will usually consider each $\mathfrak{X}_i$ as a subspace of these direct sum spaces in the natural way. When $\mathbb{I} = \{1, 2\}$, we will use $\mathfrak{X}_1 \oplus_0 \mathfrak{X}_2$ and $\mathfrak{X}_1 \oplus_p \mathfrak{X}_2$ for these spaces, respectively. If $\mathfrak{X}$ is a Banach space with closed subspaces $\mathfrak{X}_1$ and $\mathfrak{X}_2$, the *internal* direct sum of $\mathfrak{X}_1$ and $\mathfrak{X}_2$ is only defined when $\mathfrak{X}_1 \cap \mathfrak{X}_2 = \{0\}$ and $\mathfrak{X}_1 + \mathfrak{X}_2$ is closed in $\mathfrak{X}$, in which case it is defined to be $\mathfrak{X}_1 + \mathfrak{X}_2$, and denoted by $\mathfrak{X}_1 \oplus \mathfrak{X}_2$. With any of the direct sums, we can consider a direct sum of operators. There are two cases we will use—the first where we have linear homomorphisms $T_k : \mathfrak{X}_k \to \mathfrak{Y}_k$ $(r = 1 \leq k \leq n)$, in which case $T = T_1 \oplus \cdots \oplus T_n : \bigoplus_1^n \mathfrak{X}_k \to \bigoplus_1^n \mathfrak{Y}_k$ is given by $T(x_1, \ldots, x_n) = \left(T_1(x_1), \ldots, T_n(x_n)\right)$, and the second where we have linear homomorphisms $T_k : \mathfrak{X} \to \mathfrak{Y}_k$ $(1 \leq k \leq n)$, in which case $T = T_1 \oplus \cdots \oplus T_n : \mathfrak{X} \to \bigoplus_1^n \mathfrak{Y}_k$ is given by $T(x) = \left(T_1(x), \ldots, T_n(x)\right)$. It will be clear from the context which of these is intended.

We denote the *algebraic* and *projective tensor products* of Banach spaces $\mathfrak{X}$ and $\mathfrak{Y}$ by $\mathfrak{X} \otimes \mathfrak{Y}$ and $\mathfrak{X} \hat{\otimes} \mathfrak{Y}$, respectively. These are as defined in [8, section 42]. In particular, $\mathfrak{X} \hat{\otimes} \mathfrak{Y}$ is the completion of $\mathfrak{X} \otimes \mathfrak{Y}$ with respect to the projective norm, given by

$$\|u\| = \inf\left\{ \sum_{k=1}^n \|x_k\|\|y_k\| : n \in \mathbb{N}, \{x_k\}_1^n \subseteq \mathfrak{X}, \{y_k\}_1^n \subseteq \mathfrak{Y}, u = \sum_{k=1}^n x_k \otimes y_k \right\}.$$

If $\mathfrak{X}_1$, $\mathfrak{Y}_1$ are closed subspaces of $\mathfrak{X}$, $\mathfrak{Y}$ respectively, then the natural injection $\mathfrak{X}_1 \hat{\otimes} \mathfrak{Y}_1 \hookrightarrow \mathfrak{X} \hat{\otimes} \mathfrak{Y}$ has norm 1, but need not be an isometry. If $T_r : \mathfrak{X}_r \to \mathfrak{Y}_r$ $(r = 1, 2)$ are continuous linear mappings, then we have a continuous linear mapping $T_1 \otimes T_2 : \mathfrak{X}_1 \hat{\otimes} \mathfrak{X}_2 \to \mathfrak{Y}_1 \hat{\otimes} \mathfrak{Y}_2$ defined by $(T_1 \otimes T_2)(x_1 \otimes x_2) = (T_1(x_1)) \otimes (T_2(x_2))$ and extended by linearity and continuity. Then $\|T_1 \otimes T_2\| \leq \|T_1\|\|T_2\|$, and if each T_r has range dense in $\mathfrak{Y}_r$, then $T_1 \otimes T_2$ has range dense in $\mathfrak{Y}_1 \hat{\otimes} \mathfrak{Y}_2$. If $\langle X, \lambda\rangle$ and $\langle Y, \mu\rangle$ are measure spaces, then $L^1(X, \lambda) \hat{\otimes} L^1(Y, \mu)$ is isometrically isomorphic to $L^1(X \times Y, \lambda \times \mu)$, via the natural identification $(f \otimes g)(x, y) = f(x)g(y)$ $(f \in L^1(X, \lambda),\, g \in L^1(Y, \mu),\, x \in X,\, y \in Y)$.

The notation for specific normed and Banach spaces is standard, with the possible exceptions that if X is a locally compact space, then $C_{00}(X)$ will denote the space of continuous functions $X \to \mathbb{C}$ with compact support and $C_b(X)$ will denote the space of bounded continuous functions $X \to \mathbb{C}$. When X is discrete, these are $c_{00}(X)$ and $\ell^\infty(X)$, respectively.

The following is a summary of the Banach algebra theory and notation we will be using in the sequel. Again, we assume that the underlying field is $\mathbb{C}$. A *unital* Banach algebra $\mathfrak{A}$ is one with a unit, denoted $e_{\mathfrak{A}}$ or e, and if $\mathfrak{A}$ is a non-unital, the *unitization* of $\mathfrak{A}$ is $\mathfrak{A} \times \mathbb{C}$ with the product $(a_1, z_1) \cdot (a_1, z_2) = (a_1 a_2 + z_1 a_2 + z_2 a_1, z_1 z_2)$. This will be denoted $\mathfrak{A}^\sharp$. An *approximate left identity* (respectively *approximate right identity*) for $\mathfrak{A}$ is a net $\{e_n\}_{n \in \Delta}$ such that for each $a \in \mathfrak{A}$, $e_n a \to a$ (respectively $a e_n \to a$). An *approximate identity* is a net $\{e_n\}_{n \in \Delta}$ that is both an approximate left identity and an approximate right identity. We will mostly be concerned with *bounded* approximate identities. If $\mathfrak{A}$ is a Banach algebra, we denote by $\mathfrak{A}^{\mathrm{op}}$ the algebra $\mathfrak{A}$ with reversed product $a \times b = ba$ $(a, b \in \mathfrak{A})$. If $\{\mathfrak{A}_i\}_{i \in \mathbb{I}}$ is a family of Banach algebras, then $\bigoplus_{i \in \mathbb{I}}{}_0\, \mathfrak{A}_i$ and $\bigoplus_{i \in \mathbb{I}}{}_p\, \mathfrak{A}_i$ are both Banach algebras, with product defined pointwise. These are commutative if and only if each $\mathfrak{A}_i$ is commutative, and unital if and only if each $\mathfrak{A}_i$ is unital and $\mathbb{I}$ is finite. If $\mathfrak{A}_1$, $\mathfrak{A}_2$ are closed subalgebras of a Banach algebra $\mathfrak{A}$, then $\mathfrak{A}_1 \oplus \mathfrak{A}_2$, the internal direct sum of $\mathfrak{A}_1$ and $\mathfrak{A}_2$, is defined when $\mathfrak{A}_1 \oplus \mathfrak{A}_2$ is defined as a direct sum of Banach spaces, and $x_1 x_2 = x_2 x_1 = 0$ for all $x_1 \in \mathfrak{A}_1$, $x_2 \in \mathfrak{A}_2$.

If $\mathfrak{A}$ is a commutative Banach algebra, we call the space of nonzero homomorphisms $\mathfrak{A} \to \mathbb{C}$ the *carrier space* of $\mathfrak{A}$, and denote this $\Phi_{\mathfrak{A}}$. Then $\Phi_{\mathfrak{A}} \subseteq \mathfrak{A}^*$, and with the relative weak* topology, $\Phi_{\mathfrak{A}}$ is locally compact. If $\mathfrak{A} = \bigoplus_{0,\, i\in I} \mathfrak{A}_i$ or $\mathfrak{A} = \bigoplus_{p,\, i\in I} \mathfrak{A}_i$ then $\Phi_{\mathfrak{A}}$ is naturally identified with $\bigcup_{i\in I} \Phi_{\mathfrak{A}_i}$. The *Gel'fand transform* of an element $a \in \mathfrak{A}$ is denoted $\hat{a} \in C_0(\Phi_{\mathfrak{A}})$, and is given by $\hat{a}(\varphi) = \langle a, \varphi\rangle$. The Gel'fand transform is a homomorphism $\mathfrak{A} \to C_0(\Phi_{\mathfrak{A}})$. We say $\mathfrak{A}$ is *semisimple* if the Gel'fand transform is a monomorphism. If $S \subseteq \Phi_{\mathfrak{A}}$, the *kernel* of X is $\mathcal{I}(S) = \{a \in \mathfrak{A} : \hat{a}(S) = \{0\}\} = \bigcap_{\varphi\in S} \ker\varphi = {}^{\perp}S$. Clearly $\mathcal{I}(S)$ is a closed ideal of $\mathfrak{A}$. If $\mathfrak{X} \subseteq \mathfrak{A}$, then the *hull* of $\mathfrak{X}$ is $Z(\mathfrak{X}) = \{\varphi \in \Phi_{\mathfrak{A}} : \varphi(\mathfrak{X}) = \{0\}\} = \Phi_{\mathfrak{A}} \cap \mathfrak{X}^{\perp}$. Since $\mathfrak{X}^{\perp}$ is weak* closed, $Z(\mathfrak{X})$ is a closed subset of $\mathfrak{X}$. If $S \subseteq \Phi_{\mathfrak{A}}$ is the hull of some $\mathfrak{X} \subseteq \mathfrak{A}$, then we say that S is *a hull* in $\Phi_{\mathfrak{A}}$, in which case $S = Z\mathcal{I}(S)$. The *hull-kernel* topology on $\Phi_{\mathfrak{A}}$ is the topology whose closed sets are the hulls in $\Phi_{\mathfrak{A}}$. In this topology, closure is given by $E \mapsto Z\mathcal{I}(E)$. A commutative Banach algebra is *regular* if the hull-kernel topology on $\Phi_{\mathfrak{A}}$ coincides with the relative weak* topology. A hull $S \subseteq \Phi_{\mathfrak{A}}$ is a set of *synthesis* (or *spectral synthesis*) if $\mathcal{I}(S)$ is the only closed ideal of $\mathfrak{A}$ whose hull is S.

If $\mathfrak{A}$ is a Banach algebra, a left (respectively right) Banach $\mathfrak{A}$-module is a Banach space $\mathfrak{X}$ that is also a left (respectively right) $\mathfrak{A}$-module and such that the bilinear form $(a, x) \mapsto a \cdot x$ (respectively $(x, a) \mapsto x \cdot a$) is continuous. We call this bilinear form the left (respectively right) module multiplication. A Banach $\mathfrak{A}$-bimodule is an $\mathfrak{A}$-bimodule that is both a left and a right Banach $\mathfrak{A}$-module. If $\mathfrak{X}$ is a left Banach $\mathfrak{A}$-module and $\mathfrak{Y}$ is a right Banach $\mathfrak{A}$-module, $\mathfrak{X}^*$ to be a right Banach $\mathfrak{A}$-module and $\mathfrak{Y}^*$ to be a left Banach $\mathfrak{A}$-module, and $\mathfrak{X} \hat{\otimes} \mathfrak{Y}$ is a Banach $\mathfrak{A}$-bimodule, where we define the module multiplications on these by :

$$\langle x, f \cdot a\rangle = \langle a \cdot x, f\rangle \qquad (a \in \mathfrak{A}, x \in \mathfrak{X}, f \in \mathfrak{X}^*),$$

$$\langle y, a \cdot g\rangle = \langle y \cdot a, g\rangle \qquad (a \in \mathfrak{A}, y \in \mathfrak{Y}, g \in \mathfrak{Y}^*),$$

$$\text{and } (x \otimes y) \cdot a = x \otimes (y \cdot a),$$

$$a \cdot (x \otimes y) = (a \cdot x) \otimes y \qquad (a \in \mathfrak{A}, x \in \mathfrak{X}, y \in \mathfrak{Y}).$$

In particular, if $\mathfrak{X}$ is a Banach $\mathfrak{A}$-bimodule, $\mathfrak{X}^*$ is also a Banach $\mathfrak{A}$-bimodule. Such a bimodule we call a *dual* Banach $\mathfrak{A}$-bimodule. A left $\mathfrak{A}$-module morphism is a linear morphism ψ between left $\mathfrak{A}$-modules $\mathfrak{X}$ and $\mathfrak{Y}$ such that $\psi(a \cdot x) = a \cdot x$. Similarly we define right $\mathfrak{A}$-module morphisms and $\mathfrak{A}$-bimodule morphisms. There is a natural $\mathfrak{A}$-bimodule morphism $\pi : \mathfrak{A} \hat{\otimes} \mathfrak{A} \to \mathfrak{A}$ determined by putting $\pi(a \otimes b) = ab$, and extending by linearity and continuity. The mapping π can also be viewed as a left $\mathfrak{A} \otimes \mathfrak{A}^{\mathrm{op}}$-module morphism $\mathfrak{A} \hat{\otimes} \mathfrak{A}^{\mathrm{op}} \to \mathfrak{A}$, where we define the left module product by $(a \otimes b) \cdot c = abc$ $(a, b, c \in \mathfrak{A})$, extended by linearity and continuity. It follows that $\ker \pi$ is a closed left ideal of $\mathfrak{A} \hat{\otimes} \mathfrak{A}^{\mathrm{op}}$, which we call the *diagonal ideal* of $\mathfrak{A}$.

Let G be a group. We will notate the group product on G either additively or multiplicatively, depending on whether the groups we are considering are assumed to be Abelian or not. It will be stated in the text if additive notation is to be assumed. In either case, e will be used for the identity element of G. (Except when we are considering specific groups, such as $\mathbb{R}$, $\mathbb{Z}$, which have unit 0, and $\mathbb{T}$, which has unit 1. If $x \in G$, we define the function of *left* (respectively *right*) *translation* by x to be the function $G \to G$ given by ${}_x\tau(y) = xy$ (respectively $\tau_y(x) = xy$). If $f : G \to X$ is a function into some set X, we define the left and right translates of f by x to be ${}_xf = f \circ {}_{x^{-1}}\tau$ and $f_x = f \circ \tau_x$, respectively. Suppose $\mathfrak{X}$ is a set of functions on G. We say $\mathfrak{X}$ is *left* (respectively *right*) *invariant* if ${}_xf \in \mathfrak{X}$ (respectively $f_x \in \mathfrak{X}$), for each $f \in \mathfrak{X}$ and each $x \in G$. If $\mathfrak{X}$ is both left and right invariant, we say that $\mathfrak{X}$ is *bilaterally invariant* (or *invariant*). Now, if Ψ is a function from $\mathfrak{X}$ into any set S, and $\mathfrak{X}$ is left (respectively right) invariant, we say that Ψ is *left* (respectively *right*) *invariant* if $\Psi(f) = \Psi({}_xf)$ (respectively $\Psi(f) = \Psi(F_x)$), for each $f \in \mathfrak{X}$ and each $x \in G$. If $\mathfrak{X}$ is invariant, an *(bilaterally) invariant* function on $\mathfrak{X}$ is one that is both left and right invariant. If G is a locally compact group, λ_G (or λ) will denote the left Haar measure on G, that is, the unique (up to multiplication by a positive constant) left-invariant positive linear functional on $C_{00}(G)$, and Δ_G will denote the modular function on G, which is the unique homomorphism $G \to \mathbb{R}^+$ such that $\lambda_G(Xx) = \Delta_G(x)\lambda_G(X)$ for each Borel $X \subseteq G$ and each $x \in G$. We consider $L^1(G) \subseteq M(G) = C_0^*(G)$ via $\langle \psi, f \rangle = \int_G \psi(x) f(x)\, d\lambda_G(x)$ $(\psi \in C_0(G))$.

If $f, g : G \to \mathbb{C}$ are equal λ_G-almost everywhere, then so are each of the pairs ${}_xf$, ${}_xg$ and f_x, g_x. Hence, for $1 \leq p \leq \infty$, we can define the left and right translates of $f \in L^p(G)$, and we can apply the concepts of left/right/bilaterally invariant sets $\mathfrak{X} \subseteq L^p(G)$ and left/right/bilaterally invariant functions with domain $\mathfrak{X}$.

The spaces of bounded left uniformly continuous functions, right uniformly continuous functions and uniformly continuous functions $G \to \mathbb{C}$ we denote $UC_l(G)$, $UL_r(G)$ and $UC(G)$, respectively.

We consider convolution multiplication on $M(G)$, given by $\langle \psi, \mu_1 * \mu_2 \rangle = \int_G \int_G \psi(xy)\, d\mu_1(x)\, d\mu_2(y)$ by which $M(G)$ is a Banach algebra and $L^1(G)$ is a closed ideal of $M(G)$. If H is a closed normal subgroup of G, then $\tilde{T}_H$ will denote the map $M(G) \to M(G/H)$ given by $\langle \psi, \tilde{T}_H(\mu) \rangle = \int_G \psi \circ Q_H \, d\mu$ $(\psi \in C_0(G/H))$. Note that here, $\psi \circ Q_H$ is bounded and continuous, but may not lie in $C_0(G)$. However, since μ is finite and regular, the integral is defined. As discussed in [32], $\tilde{T}_H$ is an algebra epimorphism with $\|\tilde{T}_H\| \leq 1$. Also, $T_H = \tilde{T}_H|_{L^1(G)}$ maps $L^1(G)$ onto $L^1(G/H)$.

If G is a locally compact Abelian group, we have a dual group $\hat{G} = \Gamma$, the set of all continuous homomorphisms $G \to \mathbb{T}$, with the topology of uniform convergence on compact subsets of G. We will use $\langle x, \gamma \rangle$ for $\gamma(x)$ $(x \in G, \gamma \in \Gamma)$, and since we can identify $\hat{\Gamma}$ with G, we may also denote this $\langle \gamma, x \rangle$. Throughout, whenever a locally compact Abelian group G (or G_1, G_2, G', ...) is specified, it is assumed that G has dual, which will be denoted Γ (or Γ_1, Γ_2, Γ', ...). Conversely, if we specify a locally compact Abelian group Γ it is assumed that Γ has dual G. The Banach algebras $L^1(G)$ and $M(G)$ are commutative and semisimple, and $L^1(G)$ is regular. The map $\Gamma \to L^\infty(G)$ given by $\gamma \mapsto \overline{\gamma}$ maps homeomorphically onto $\Phi_{L^1(G)}$, and in this sense, we identify Γ with $\Phi_{L^1(G)}$. The Gel'fand transform on $L^1(G)$ is thus identified with the Fourier transform $f \mapsto \hat{f}$, where $\hat{f}(\gamma) = \int_G f(x)\overline{\langle x, \gamma \rangle}\, d\lambda_G(x)$. Similarly, if we consider $C_b(G)$ as a subspace of $M^*(G)$ via $\langle \mu, F \rangle = \int_G F\, d\mu$, then we obtain an injection $\Gamma \to \Phi_{M(G)}$. If we thus identiry Γ with a subset of $\Phi_{M(G)}$, then the weak* topology from $M(G)^*$, when restricted to Γ, is certainly no weaker than the group topology, and since each $\mu \in M(G)$ has $\hat{\mu}$ a continuous function on Γ, the topologies coincide. The Fourier-Stieltjes transform is then obtained by restricting the Gel'fand transform of a measure $\mu \in M(G)$ to Γ. In this case, the notation $\hat{\mu}$ will denote the Fourier-Stieltjes transform, rather than the Gel'fand transform, unless otherwise stated. Put $A(\Gamma) = L^1(G)\widehat{\ }$ and $B(\Gamma) = M(G)\widehat{\ }$, each with pointwise multiplication and norms given by $\|\hat{f}\|_{A(\Gamma)} = \|f\|_{L^1(G)}$ and $\|\hat{\mu}\|_{B(\Gamma)} = \|\mu\|_{M(G)}$. We call these the *Fourier* and *Fourier-Stieltjes* algebras on Γ, respectively. Then $A(\Gamma)$ and $B(\Gamma)$ are algebras of functions isometrically isomorphic to $L^1(G)$ and $M(G)$. We have $L^1(G) \subseteq C_0(\Gamma)$ and $M(G) \subseteq UC(\Gamma)$, and if we denote Γ with its discrete topology by Γ_d, then $B(\Gamma) = B(\Gamma_d) \cap C(\Gamma)$, and $\|F\|_{B(\Gamma)} = \|F\|_{B(\Gamma_d)}$ $(F \in B(\Gamma))$.

Let H be a closed subgroup of G. Then H is open if and only if H is clopen. The *index* of H in G is $[G:H] = |G/H|$. The *annihilator* of H (in Γ) is $\mathrm{Ann}_\Gamma(H) = \{\, \gamma \in \Gamma : \langle x, \gamma \rangle = 1 \ (x \in H) \,\}$, a closed subgroup of Γ. We have that $\mathrm{Ann}_G(\mathrm{Ann}_\Gamma(H)) = H$ and H is compact if and only if $\Lambda = \mathrm{Ann}_\Gamma(H)$ is clopen. We identify $\hat{H}$ with Γ/Λ and consequently $(G/H)\widehat{\ }$ with Λ. Then the epimorphism $\tilde{T}_H : M(G) \to M(G/H)$ corresponds to $\rho_\Lambda : B(\Gamma) \to B(\Lambda)$, so that $B(\Lambda) = \rho_\Lambda(B(\Gamma))$, which we denote $B(\Gamma)|_\Lambda$. Similarly, $T_H : L^1(G) \to L^1(G/H)$ is an epimorphism, so $A(\Lambda) = \rho_\Lambda(A(\Gamma)) = A(\Gamma)|_\Lambda$. We call a set $E \subseteq G$ a *coset* in G if $E = \tau_x(H)$, for some $x \in G$, and some subgroup H of G. A *subcoset* of a coset E is a coset that is a subset of E. Since τ_x is a homeomorphism, E is closed (respectively compact, clopen) if and only if H is closed (respectively compact, clopen). A nonempty set $E \subseteq G$ is a coset if and only if $E \cdot E \cdot E^{-1} \subseteq E$, then with $x \in E$ and $H = E \cdot E^{-1}$, H is a subgroup and $E = Hx$. If E is a closed coset in a locally compact Abelian group Γ, say $E = \tau_x(\Lambda)$, define the Banach algebras

$$A(E) = \{\, f \in C_0(E) : f \circ \tau_x \in A(\Lambda) \,\} \text{ and } B(E) = \{\, F \in C(E) : F \circ \tau_x \in B(\Lambda) \,\},$$

with pointwise product, $\|f\|_{A(E)} = \|f \circ \tau_x\|_{A(\Lambda)}$, and $\|F\|_{B(E)} = \|F \circ \tau_x\|_{B(\Lambda)}$. Clearly these are Banach algebras isometrically isomorphic to $A(\Lambda)$ and $B(\Lambda)$, respectively, and we can identify E with $\Phi_{A(E)}$. Consequently, we will be able to take any result concerning the Banach-algebraic properties of the group and measure algebras, and apply it to these algebras. In particular, $A(\Gamma)$ and $B(\Gamma)$ are translation-invariant, so that $A(E) = A(\Gamma)|_E$ and $B(E) = B(\Gamma)|_E$.

If $\{G_i\}_{i\in I}$ is a family of locally compact Abelian groups, their *product* is the group $\prod_{i\in I} G_i$ with the product topology. In the case $i = \{1,2\}$, we denote this $G_1 \times G_2$. The *direct sum* of $\{G_i\}_{i\in I}$, denoted $\sum_{i\in I} G_i$, consists of those $\{x_i\} \in \prod_{i\in I} G_i$ such that all but finitely many x_i are e. We will only consider such a group in the case where each G_i is discrete, in which case $\sum_{i\in I} G_i$ is discrete. If H_1 and H_2 are closed subgroups of a locally compact Abelian group G, then $H_1 \oplus H_2$ is defined to be $H_1 H_2$ only when $H_1 \cap H_2 = \{e\}$ and the isomorphism $H_1 \times H_2 \to H_1 H_2$ given by $(x, y) \mapsto xy$ is a homeomorphism.

The numbering in this thesis is on three levels, the first part being a numeral indicating a chapter or a letter indicating an appendix. The second number indicates the section within the relevant chapter or appendix, and the third number indicates the subsection, which will consist of a single theorem, proposition, lemma, definition, or example. Subsections are numbered consecutively through a section, so that there will not be a Theorem 1 and a Proposition 1 in the same section. All numbers will be cited in any cross referencing, whether within a section or between chapters.

0.2. Amenability and Property (G)

Suppose G is a locally compact group and $\mathfrak{X}$ is a subspace of $L^\infty(G)$. A *mean* on $\mathfrak{X}$ is a linear functional M on $\mathfrak{X}$ such that for each $f \in \mathfrak{X}$ with $\operatorname{rng} f \subseteq \mathbb{R}$, $\operatorname{ess\,inf}_{x\in G} f(x) \leq M(f) \leq \operatorname{ess\,sup}_{x\in G} f(x)$. (If $\mathfrak{X}$ is a subspace of $C_b(G) \subseteq L^\infty(G)$, we may replace "ess inf" and "ess sup" by "inf" and "sup".) Note that if we give $\mathfrak{X}$ the norm from $L^\infty(G)$, then a mean is continuous with $\|M\| \leq 1$. If $1 \in \mathfrak{X}$, then $M \in \mathfrak{X}^*$ is a mean if and only if $\|M\| \leq 1$ and $M(1) = 1$.

We call a group *amenable* if there exists a left-invariant mean on $L^\infty(G)$. Equivalent conditions are the existence of right-invariant means and invariant means on any of the spaces $C_b(G)$, $UC(G)$, $UC_r(G)$, or $UC_l(G)$. (See, for example, [17, 30, or 33].) Another useful type of condition equivalent to amenability are the Følner conditions. (See [30, Chapter 4 or 33, Section 2.7].) The particular Følner condition we will consider here is the existence of a *summing net* for G, that is, a net $\{K_n\}_{n\in\Delta}$ of compact subsets such that $\bigcup_{n\in\Delta} K_n = G$ and for each compact set U,

$$\sup_{u\in U} \frac{\lambda(K_n \triangle K_n u)}{\lambda(K_n)} \to 0.$$

This is shown to be equivalent to the above characterizations of amenability in [30, Theorem 4.16].

Locally compact Abelian groups are amenable, as are compact groups. An example of a non-amenable group is F_2, the free group on two generators.

We now consider the property of amenability in Banach algebras. Many of these basic results originated in the papers [23, 24] of Johnson, and a description can be found in Sections 43 and 44 of [8]. Suppose $\mathfrak{A}$ is a Banach algebra and $\mathfrak{X}$ is a Banach $\mathfrak{A}$-bimodule. A linear homomorphism $D : \mathfrak{A} \to \mathfrak{X}$ is called a *derivation* if $D(ab) = D(a)\cdot b + a \cdot D(b)$. If $x \in X$, the map $D_x : \mathfrak{A} \to \mathfrak{X}$ given by $D_x(a) = a\cdot x - x\cdot a$ is a derivation. Derivations of this type we call *inner*. We say that $\mathfrak{A}$ is *amenable* if any derivation into a dual $\mathfrak{A}$-bimodule is inner.

This is based on a topological version of the Hochschild cohomology of non-topological associative algebras. Let $\mathfrak{X}$ be a Banach $\mathfrak{A}$-bimodule and let $\mathcal{B}^n(\mathfrak{A},\mathfrak{X})$ be the space of bounded n-linear mappings $\mathfrak{A}^n \to \mathfrak{X}$. (We identify $\mathcal{B}^0(\mathfrak{A},\mathfrak{X})$ with $\mathfrak{X}$.) Now define $\delta_n : \mathcal{B}^{n-1}(\mathfrak{A},\mathfrak{X}) \to \mathcal{B}^n(\mathfrak{A},\mathfrak{X})$ by

$$\begin{aligned}\delta^n(T)(a_1,\ldots,a_n) = a_1\cdot T(a_2,\ldots,a_n) &+ \sum_{j=1}^{n-1}(-1)^j T(a_1,\ldots,a_j a_{j+1},\ldots,a_n)\\ &+ (-1)^n T(a_1,\ldots,a_{n-1})\cdot a_n\end{aligned}$$

where $T \in \mathcal{B}^{n-1}(\mathfrak{A},\mathfrak{X})$ and $a_1,\ldots,a_n \in \mathfrak{A}$. A standard property of these mappings is that $\delta^{n+1}\circ\delta^n = 0$, so that we have the complex

$$0 \xrightarrow{\delta_0} \mathfrak{X} \xrightarrow{\delta_1} \mathcal{B}(\mathfrak{A},\mathfrak{X}) \xrightarrow{\delta_2} \mathcal{B}^2(\mathfrak{A},\mathfrak{X}) \xrightarrow{\delta_3} \cdots$$

The n^{th} cohomology group of this complex is $H^n(\mathfrak{A},\mathfrak{X}) = \ker\delta^{n+1}/\operatorname{rng}\delta^n$. In particular, if $x \in \mathfrak{X}$, then $\delta^1(x)(a) = a\ x - x\cdot a$ and if $x \in \mathcal{B}^1(\mathfrak{A},\mathfrak{X})$, then

$\delta^2(T)(a_1, a_2) = a_1T(a_2) - T(a_1a_2) + T(a_1)a_2$, so that $\mathfrak{A}$ is amenable if and only if $H^1(\mathfrak{A}, \mathfrak{X}^*) = 0$ for each dual Banach $\mathfrak{A}$-bimodule $\mathfrak{X}^*$. It can be shown, using standard "dimension-reducing" techniques, that in this case we have $H^n(\mathfrak{A}, \mathfrak{X}^*) = 0$, for all n.

The use of the term "amenable" for such algebras originated in the paper [23] of B.E. Johnson, and is motivated by the following theorem.

0.2.1. Theorem. *[23, Theorem 2.5] Suppose G is a locally compact group. Then G is amenable if and only if $L^1(G)$ is amenable.*

Again, there are several equivalent characterizations of amenability in Banach algebras. For these, we make some further definitions.

Let $\mathfrak{A}$ be a Banach algebra. An *approximate diagonal* is a bounded net $\{d_n\}_{n\in\Delta}$ in $\mathfrak{A} \hat{\otimes} \mathfrak{A}$ such that $\{\pi(d_n)\}_{n\in\Delta}$ is a bounded approximate left identity for $\mathfrak{A}$ and for each $a \in \mathfrak{A}$, $a \cdot d_n - d_n \cdot a \to 0$. A *virtual diagonal* is an element $d \in (\mathfrak{A} \hat{\otimes} \mathfrak{A})^{**}$ such that for each $a \in \mathfrak{A}$, $\pi^{**}(d) \cdot a = a$ and $d \cdot a = a \cdot d$. A *diagonal* is an element $d \in \mathfrak{A} \hat{\otimes} \mathfrak{A}$ such that $\pi(d) = e \in \mathfrak{A}$ and $da = ad$. (This is called a *splitting idempotent* in [27], in reference to the cohomological implications of the existence of such an element.) We now have the following theorem, obtained by combining Lemma 1.2 and Theorem 1.3 of the paper [24] of B.E. Johnson with [13, Theorem 3.10], a result originally due to A.Ya. Khelemskii.

0.2.2. Theorem. *Suppose $\mathfrak{A}$ is a Banach algebra, then the following are equivalent :*

(i) $\mathfrak{A}$ is amenable,

(ii) there is an approximate diagonal for $\mathfrak{A}$,

(iii) there is a virtual diagonal for $\mathfrak{A}$, and

(iv) the diagonal ideal for $\mathfrak{A}$, $\ker\pi \subseteq \mathfrak{A}\hat{\otimes}\mathfrak{A}^{\mathrm{op}}$, has a bounded approximate identity.

In the case where $\mathfrak{A}$ is finite-dimensional, the existence of a virtual diagonal is clearly equivalent to the existence of a diagonal. By results of G. Hochschild [21, Theorem 4.1], and M.J. Liddell [27, Theorem 1.3], we have the following.

0.2.3. Proposition. *Suppose $\mathfrak{A}$ is a finite-dimensional complex algebra, then the following are equivalent :*

(i) $\mathfrak{A}$ is amenable,

(ii) there is a diagonal for $\mathfrak{A}$, and

(iii) $\mathfrak{A}$ is isomorphic to a finite direct sum of matrix algebras $M_n(\mathbb{C})$.

Examples of amenable Banach algebras other than the group algebras of amenable groups are $C_0(X)$, for X a locally compact topological space, and $\mathcal{K}(H)$, the algebra of compact operators on a Hilbert space H. These are shown to be amenable using Theorem 0.2.1 and the following proposition.

0.2.4. Proposition. *[23, 5.3] Suppose $\mathfrak{A}$ and $\mathfrak{B}$ are amenable Banach algebras and $\nu : \mathfrak{A} \to \mathfrak{B}$ is a continuous algebra homomorphism with range dense in $\mathfrak{B}$. If $\mathfrak{A}$ is amenable, then $\mathfrak{B}$ is amenable.*

Define a Banach algebra $\mathfrak{A}$ to have *property* (G) if there exists an amenable locally compact group G and a continuous homomorphism $\nu : L^1(G) \to \mathfrak{A}$ with range dense in $\mathfrak{A}$. Clearly property (G) is sufficient for amenability. It is natural to ask whether all amenable Banach algebra have property (G).

We present some basic results on property (G) which will aid us in later sections, when we will characterize property (G) for certain Banach algebras $\mathfrak{A}$.

0.2.5. Proposition. *Suppose $\mathfrak{A}$ and $\mathfrak{B}$ are Banach algebras with property* (G), *then $\mathfrak{A} \oplus \mathfrak{B}$ and $\mathfrak{A} \hat{\otimes} \mathfrak{B}$ have property* (G).

Proof. By hypothesis, there exist amenable locally compact groups G_1 and G_2 and continuous homomorphisms $\nu_1 : L^1(G_1) \to \mathfrak{A}$ and $\nu_2 : L^1(G_2) \to \mathfrak{B}$ with $\overline{\operatorname{rng} \nu_1} = \mathfrak{A}$ and $\overline{\operatorname{rng} \nu_2} = \mathfrak{B}$. Then $\nu_1 \oplus \nu_2 : L^1(G_1) \oplus L^1(G_2) \to \mathfrak{A} \oplus \mathfrak{B}$ and $\nu_1 \otimes \nu_2 : L^1(G_1) \hat{\otimes} L^1(G_2) \to \mathfrak{A} \hat{\otimes} \mathfrak{B}$ are dense-ranged continuous homomorphisms, so it suffices to show that $L^1(G_1) \oplus L^1(G_2)$ and $L^1(G_1) \hat{\otimes} L^1(G_2)$ have property (**G**). The groups $G_1 \times G_2$ and $G_1 \times G_2 \times \mathbb{Z}_2$ are amenable with

$$L^1(G_1 \times G_2) \cong L^1(G_1) \hat{\otimes} L^1(G_2),$$

$$\begin{aligned} L^1(G_1 \times G_2 \times \mathbb{Z}_2) &\cong L^1(G_1 \times G_2) \hat{\otimes} \mathbb{C}^2 \\ &\cong L^1(G_1 \times G_2) \oplus L^1(G_1 \times G_2), \end{aligned}$$

and $T_{G_2} \oplus T_{G_1} : L^1(G_1 \times G_2) \oplus L^1(G_1 \times G_2) \to L^1(G_1) \oplus L^1(G_2)$ is an epimorphism. ■

[illegible] Theorem 4.1], and [illegible]

0.2.3. Proposition. *Suppose* $\mathfrak{A}$ *is a finite-dimensional complex algebra, then the following are equivalent:*

(i) $\mathfrak{A}$ *is amenable;*

(ii) there is a diagonal for $\mathfrak{A}$*; and*

(iii) $\mathfrak{A}$ *is isomorphic to a finite direct sum of matrix algebras* $M_n(\mathbb{C})$*.*

Examples of amenable Banach algebras other than the group algebras of amenable groups are $C_0(X)$, for X a locally compact topological space, and $K(H)$, the algebra of compact operators on a Hilbert space H. These are shown to be amenable using Theorem 0.2.1 and the following proposition.

0.2.4. Proposition. [O?, 5.3] *Suppose* $\mathfrak{A}$ *and* $\mathfrak{B}$ *are amenable Banach algebras and* $\nu : \mathfrak{A} \to \mathfrak{B}$ *is a continuous algebra homomorphism with range dense in* $\mathfrak{B}$*. If* $\mathfrak{A}$ *is amenable, then* $\mathfrak{B}$ *is amenable.*

Define a Banach algebra $\mathfrak{A}$ to have *property (G)* if there exists an amenable locally compact group G and a continuous homomorphism $\nu : L^1(G) \to \mathfrak{A}$ with range dense in $\mathfrak{A}$. Clearly property (G) is sufficient for amenability. It is natural to ask whether all amenable Banach algebras have property (G).

We present some basic results on property (G) which will aid us in later sections, where we will characterize property (G) for certain Banach algebras $\mathfrak{A}$.

0.2.5. Proposition. *Suppose* $\mathfrak{A}$ *and* $\mathfrak{B}$ *are Banach algebras with property* (G), *then* $\mathfrak{A} \oplus \mathfrak{B}$ *and* $\mathfrak{A} \hat{\otimes} \mathfrak{B}$ *have property* (G).

Proof. By hypothesis, there exist amenable locally compact groups G_1 and G_2 and continuous homomorphisms $\nu_1 : L^1(G_1) \to \mathfrak{A}$ and $\nu_2 : L^1(G_2) \to \mathfrak{B}$ with $\overline{\operatorname{ran} \nu_1} = \mathfrak{A}$ and $\overline{\operatorname{ran} \nu_2} = \mathfrak{B}$. Then $\nu_1 \oplus \nu_2 : L^1(G_1) \oplus L^1(G_2) \to \mathfrak{A} \oplus \mathfrak{B}$ and $\nu_1 \hat{\otimes} \nu_2 : L^1(G_1) \hat{\otimes} L^1(G_2) \to \mathfrak{A} \hat{\otimes} \mathfrak{B}$ are dense-ranged continuous homomorphisms,

Chapter 1. Property (G) in Commutative Banach Algebras

In the present chapter, we will investigate property (G) for certain families of commutative amenable Banach algebras. As one may expect, to assess property (G) in commutative Banach algebras, it is sufficient to consider only homomorphisms $L^1(G) \to \mathfrak{A}$, for G an *Abelian* locally compact group.

The actual Banach algebras $\mathfrak{A}$ we consider are themselves subalgebras of commutative group algebras, enabling the use of a result of P.J. Cohen on homomorphisms between commutative group algebras. Results obtained below on such homomorphisms seem to be of independent interest, and suggest several possible generalizations that will be followed in later chapters.

1.1. Reduction to Abelian Groups

The sole result of this section is that to ascertain property (G) for commutative group algebras, it is enough to consider Abelian groups. If $\mathfrak{A}$ is a Banach algebra, a *commutator* of $\mathfrak{A}$ is an element of the form $ab - ba$, for some $a, b \in \mathfrak{A}$. The *commutator ideal* of $\mathfrak{A}$ is defined to be the closed ideal generated by the commutators in $\mathfrak{A}$. It is clear that the commutator ideal is the smallest closed ideal $\mathcal{I}$ for which the quotient algebra $\mathfrak{A}/\mathcal{I}$ is commutative.

If G is a locally compact group, a *commutator* of G is an element of the form $xyx^{-1}y^{-1}$, for some $x, y \in G$. The *commutator subgroup* is the closed subgroup of G generated by the commutators in G. The commutator subgroup of G is a closed normal subgroup, and it is the smallest closed normal subgroup C for which the quotient group G/C is Abelian. (See [20, Theorem 23.8].)

1.1.1. Lemma. *Suppose G is a locally compact group with commutator subgroup C and T_C is the natural epimorphism $L^1(G) \to L^1(G/C)$, then $\ker T_C$ is the commutator ideal of $L^1(G)$.*

Proof. Let $\mathcal{J}$ be the commutator ideal of $L^1(G)$. The homomorphism T_C has range in a commutative Banach algebra, so any commutator of $\mathfrak{A}$ is contained in $\ker T_C$. Hence $\mathcal{J} \subseteq \ker T_C$.

Conversely, by [32, 3.6.4], we have $\ker T_C = \overline{\operatorname{span}}\{f - {}_xf : x \in C, f \in L^1(G)\}$. Put $H = \{x \in G : f - {}_xf \in \mathcal{J},\ (f \in L^1(G))\}$, a closed subgroup of G. Let $\{e_n\}_{n\in\Delta}$ be a bounded approximate identity for $L^1(G)$, and let $x, y \in G$. For $n \in \Delta$, put $g_n = {}_ye_n * {}_xe_n - {}_xe_n * {}_ye_n \in \mathcal{J}$ so that if $f \in L^1(G)$, then $g_n * f \in \mathcal{J}$ and

$$\begin{aligned}\|({}_{yx}f - {}_{xy}f) - g_n * f\| &\le \|{}_y({}_xf - e_n * {}_xf)\| + \|{}_ye_n * {}_x(f - e_n * f)\| \\ &\quad + \|{}_x({}_yf - e_n * {}_yf)\| + \|{}_xe_n * {}_y(f - e_n * f)\| \\ &\le \|{}_xf - e_n * {}_xf\| + \|{}_yf - e_n * {}_yf\| + 2\|e_n\|\|f - e_n * f\| \\ &\to 0.\end{aligned}$$

But $\mathcal{J}$ is closed and translation-invariant, so ${}_{yx}f - {}_{xy}f \in \mathcal{J}$ and ${}_{x^{-1}y^{-1}xy}f - f \in \mathcal{J}$. Hence $x^{-1}y^{-1}xy \in H$. Hence $C \subseteq H$ and $\ker T_C \subseteq \mathcal{J}$. ■

1.1.2. Proposition. *Suppose G is a locally compact group and $\nu : L^1(G) \to \mathfrak{A}$ is a continuous algebra homomorphism into a commutative Banach algebra, then there is a locally compact Abelian group G' and a continuous algebra homomorphism $\nu' : L^1(G') \to \mathfrak{A}$ with $\operatorname{rng}\nu = \operatorname{rng}\nu'$ and $\|\nu'\| = \|\nu\|$.*

Proof. Let C be the commutator subgroup of G, so that $G' = G/C$ is a locally compact Abelian group, and by Lemma 1.1.1, the commutator ideal of $L^1(G)$ is $\mathcal{J} = \ker T_C$. Clearly $\mathcal{J} \subseteq \ker\nu$. Now, by [32, 3.4.4], T_C is an epimorphism which induces an isometric isomorphism $L^1(G)/\mathcal{J} \to L^1(G/C)$. It follows that $\nu' = \nu \circ T_C^{-1} : L^1(G/C) \to \mathfrak{A}$ is a well-defined continuous algebra homomorphism, with $\|\nu'\| = \|\nu\|$. ■

Now, since locally compact Abelian groups are amenable, a commutative Banach algebra $\mathfrak{A}$ has property **(G)** if and only if there is a locally compact Abelian group G and a dense-ranged homomorphism $L^1(G) \to \mathfrak{A}$. Since the emphasis of the rest of this chapter is on property **(G)** in commutative group algebras, all groups considered herein will be Abelian. For this reason, *for all locally compact groups in the remainder of this chapter, we will use additive notation for the group product.*

1.2. Homomorphisms Between Commutative Banach Algebras

Suppose $\nu : \mathfrak{A} \to \mathfrak{B}$ is a continuous homomorphism between commutative Banach algebras. Then for $\varphi \in \Phi_{\mathfrak{B}}$, φ is a nonzero homomorphism $\mathfrak{B} \to \mathbb{C}$, so $\varphi \circ \nu = \nu^*(\varphi)$ is a homomorphism $\mathfrak{A} \to \mathbb{C}$, that is $\nu^*(\varphi) \in \Phi_{\mathfrak{A}} \cup \{0\}$. Putting $Y = \{\, \varphi \in \Phi_{\mathfrak{B}} : \nu^*(\varphi) \neq 0 \,\}$ and $\alpha = \nu^*|_Y$, we have

$$a \in \mathfrak{A} \implies \widehat{\nu(a)}(\varphi) = \begin{cases} \hat{a} \circ \alpha(\varphi) & \text{if } \varphi \in Y; \\ 0 & \text{otherwise.} \end{cases} \tag{1}$$

We will often abbreviate this to $\widehat{\nu(a)} = \hat{a} \circ \alpha$. If $\mathfrak{B}$ is semisimple, (1) serves to determine ν, and so we will mainly restrict our attention to this case. The following proposition brings together some well-known and easily obtained results concerning such an analysis of homomorphisms.

1.2.1. Proposition. *Suppose $\mathfrak{A}$ and $\mathfrak{B}$ are commutative Banach algebras and $\nu : \mathfrak{A} \to \mathfrak{B}$ is a continuous homomorphism. Defining Y and α as above, $Y = \Phi_{\mathfrak{B}} \setminus Z(\operatorname{rng} \nu)$, so that Y is an open subset of $\Phi_{\mathfrak{B}}$ and if $\mathfrak{A}$ is unital, then Y is compact. Also, α is a continuous, proper and closed map into $Z(\ker \nu)$, and if $\mathfrak{B}$ is semisimple, then $Z(\ker \nu)$ is the hull-kernel closure of $\alpha(Y)$.*

Proof. Clearly $\varphi \in Z(\operatorname{rng} \nu)$ if and only if $\varphi \circ \nu = 0$, so that $Y = \Phi_{\mathfrak{B}} \setminus Z(\operatorname{rng} \nu)$, and Y is an open subset of $\Phi_{\mathfrak{B}}$. If e is a unit in $\mathfrak{A}$, then $\nu(e)$ is an idempotent in $\mathfrak{B}$, with $\widehat{\nu(e)} = \hat{e} \circ \alpha = \chi_Y \in C_0(\Phi_{\mathfrak{B}})$, so Y is compact.

If we give both $\mathfrak{B}^*$ and $\mathfrak{A}^*$ their weak* topology, then ν^* is clearly continuous, and so it follows that α is continuous. To show α to be proper, suppose $C \subseteq \Phi_{\mathfrak{A}}$ is compact. For each $\varphi \in C$, there exists $a_\varphi \in \mathfrak{A}$ with $|\varphi(a_\varphi)| > 1$, so that $K_\varphi = \{\varphi' \in \Phi_{\mathfrak{A}} : |\hat{a}_\varphi(\varphi')| \geq 1\}$ is a closed neighbourhood of φ. Moreover, since $\hat{a}_\varphi \in C_0(\Phi_{\mathfrak{A}})$, K_φ is compact. Similarly $\alpha^{-1}(K_\varphi) = \{\varphi' \in \Phi_{\mathfrak{B}} : |\widehat{\nu(a_\varphi)}(\varphi')| \geq 1\}$ is compact. Take a finite set $\varphi_1, \ldots, \varphi_n$ such that $K_{\varphi_1}, \ldots, K_{\varphi_n}$ cover C, and then $\alpha^{-1}(C) \subseteq \alpha^{-1}(K_{\varphi_1}) \cup \cdots \cup \alpha^{-1}(K_{\varphi_n})$, which is compact, so $\alpha^{-1}(C)$ is compact. Then from [7, Section 1.10.10, Prop. 15 & Section 1.10.1, Prop. 7], or by a straightforward argument involving filters, we have that α is closed.

Clearly any $a \in \ker \nu$ has $\hat{a} \circ \alpha = 0$, so $\alpha(Y) \subseteq Z(a)$, giving $\alpha(Y) \subseteq Z(\operatorname{rng} \nu)$. If $\mathfrak{B}$ is semisimple, then $\nu(a) = 0$ if and only if $\hat{a} \circ \alpha = 0$, and so $\ker \nu = \mathcal{I}(\alpha(Y))$ and $Z(\ker \nu) = Z\mathcal{I}(\alpha(Y))$. ∎

It is worth noting at this stage that we are considering α as a map from Y, a locally compact space in its own right, into $\Phi_{\mathfrak{A}}$. It is in this capacity that α is closed, for even if Y is not a closed subset of $\Phi_{\mathfrak{B}}$, Y is a closed subset of Y, and so $\alpha(Y)$ is a closed subset of $\Phi_{\mathfrak{A}}$. Similarly for any closed subset of V of Y, $\alpha(V)$ is closed in $\Phi_{\mathfrak{A}}$, regardless of whether V is closed in $\Phi_{\mathfrak{B}}$.

Equation (1) also gives a necessary condition for an element of $\mathfrak{B}$ to be in rng ν, in that for any $b \in \operatorname{rng} \nu$, we must have $\hat{b}(\varphi) = 0$ when $\varphi \notin Y$ and $\hat{b}(\varphi_1) = \hat{b}(\varphi_2)$ when $\varphi_1, \varphi_2 \in Y$ and $\alpha(\varphi_1) = \alpha(\varphi_2)$.

1.2.2. Definition. Suppose $\mathfrak{B}$ is a commutative Banach algebra and $\sim$ is an equivalence relation on U, an open subset of $\Phi_{\mathfrak{B}}$. Define $\kappa_{\mathfrak{B}}(\sim)$, the *$\sim$-class subalgebra* of $\mathfrak{B}$ by

$$\kappa_{\mathfrak{B}}(\sim) = \{b \in \mathfrak{B} : \hat{b} = 0 \text{ off } U \text{ and } \hat{b}(\varphi_1) = \hat{b}(\varphi_2) \text{ whenever } \varphi_1 \sim \varphi_2\}.$$

The equivalence $\sim$ will usually be given by $\varphi_1 \sim \varphi_2 \iff \psi(\varphi_1) = \psi(\varphi_2)$, where ψ is a function with domain U. (Of course, every equivalence relation can be expressed in this manner.) In such a case, we will use the notation $\kappa_{\mathfrak{B}}(\psi)$ or $\kappa(\psi)$ for $\kappa_{\mathfrak{B}}(\sim)$, and $\hat{\kappa}_{\mathfrak{B}}(\psi)$ (or $\hat{\kappa}(\psi)$) will denote $\{\hat{b} : b \in \kappa_{\mathfrak{B}}(\psi)\}$. Such subalgebras are considered in the ninth chapter of [37] in the case where $\mathfrak{B}$ is the group algebra of a compact Abelian group.

1.2.3. Lemma. *With $\mathfrak{B}$, U and $\sim$ as above, $\kappa_{\mathfrak{B}}(\sim)$ is a closed subalgebra of $\mathfrak{B}$.*

Proof. Clearly $\kappa_{\mathfrak{B}}(\sim) = \bigcap_{\varphi \notin U} \ker\varphi \ \cap \ \bigcap_{\varphi_1 \sim \varphi_2} \ker(\varphi_1 - \varphi_2)$, which is a closed subspace of $\mathfrak{B}$. It is also clear that $\kappa_{\mathfrak{B}}(\sim)$ is closed under multiplication, so that $\kappa_{\mathfrak{B}}(\sim)$ is a subalgebra. ∎

We now consider the application of these ideas to an analysis of the range of an algebra homomorphism.

1.2.4. Lemma. *Suppose $\mathfrak{A}, \mathfrak{B}$ are commutative semisimple Banach algebras, $\nu : \mathfrak{A} \to \mathfrak{B}$ is a homomorphism, $Y = \Phi_{\mathfrak{B}} \setminus Z(\operatorname{rng}\nu)$, and $\alpha = \nu^*|_Y : Y \to \Phi_{\mathfrak{A}}$. For $b \in \mathfrak{B}$, we have $b \in \kappa(\alpha)$ if and only if $\hat{b} = 0$ off Y and $\hat{b} \circ \alpha^{-1} \in C_0(\alpha(Y))$; whereas $b \in \operatorname{rng}\nu$ if and only if $\hat{b} = 0$ off Y and $\hat{b} \circ \alpha^{-1} : \alpha(Y) \to \mathbb{C}$ is well-defined and has an extension in $\hat{\mathfrak{A}}$.*

Proof. Suppose $b \in \kappa(\alpha)$, so that $\hat{b} \circ \alpha^{-1}$ is a well-defined function $\alpha(Y) \to \mathbb{C}$. Let $V \subseteq \mathbb{C}$ be closed, then $\hat{b}^{-1}(V) \cap Y$ is a closed subset of Y, and since α is closed, $\alpha \circ \hat{b}^{-1}(V) = (\hat{b} \circ \alpha^{-1})^{-1}(V)$ is closed. Hence $\hat{b} \circ \alpha^{-1}$ is continuous. If we also have $0 \notin V$, then $\hat{b}^{-1}(V)$ is compact, and since α is continuous, $(\hat{b} \circ \alpha^{-1})^{-1}(V)$ is compact. Thus $\hat{b} \circ \alpha^{-1} \in C_0(\alpha(Y))$. The remaining statements are clear from definitions. ∎

It will be shown in Section 1.5 that if $\mathfrak{A}$ and $\mathfrak{B}$ are commutative group algebras, then $\operatorname{rng}\nu = \kappa(\alpha)$. To illustrate the elements needed for a proof of such a result, we consider the following much easier result.

1.2.5. Proposition. *Suppose X_1 and X_2 are locally compact topological spaces, and $\nu : C_0(X_1) \to C_0(X_2)$ is a homomorphism. Let $Y = X_2 \setminus Z(\operatorname{rng}\nu)$ and $\alpha = \nu^*|_Y : Y \to X_1$, then $\operatorname{rng}\nu = \kappa(\alpha)$.*

Proof. We have $Y \subseteq X_2$ open and $\alpha : Y \to X_1$ continuous, proper and closed, so $\alpha(Y)$ is a closed subset of X_1. Let $X_1^\infty = X_1 \cup \{\infty\}$, the one-point compactification of X_1, and define f on $\alpha(Y) \cup \{\infty\}$ to be the extension of $\hat{b} \circ \alpha^{-1}$ obtained by setting $f(\infty) = 0$. Then $\alpha(Y) \cup \{\infty\}$ is a closed subset of X_1^∞ with $f \in C(\alpha(Y) \cup \{\infty\})$, so by the Tietze Extension Theorem [7, 9.4.2], there is a function $g \in C(X_1^\infty)$ extending f. But then $g|_{X_1} \in \mathfrak{A}$ is an extension of $\hat{b} \circ \alpha^{-1}$, as required. ■

1.2.6. Corollary. *Suppose X is a locally compact topological space, $\mathfrak{A}$ is a commutative semisimple Banach algebra and $\nu : C_0(X) \to \mathfrak{A}$ is a homomorphism. Let $Y = \Phi_{\mathfrak{A}} \setminus Z(\operatorname{rng} \nu)$ and $\alpha = \nu^*|_Y : Y \to X$, then $\operatorname{rng} \nu = \kappa_{\mathfrak{B}}(\alpha)$.*

Proof. Composing ν with the Gel'fand transform on $\mathfrak{A}$ gives a homomorphism $\nu' : C_0(X) \to C_0(\Phi_{\mathfrak{A}})$. Then with $Y' = \Phi_{\mathfrak{A}} \setminus Z(\operatorname{rng} \nu')$ and $\alpha' = \nu'|_Y$, we actually have $Y = Y'$ and $\alpha = \alpha'$, so by Proposition 1.2.5, $\operatorname{rng} \nu' = \kappa_{C_0(\Phi_{\mathfrak{A}})}(\alpha)$. However, $\operatorname{rng} \nu' = (\operatorname{rng} \nu)\widehat{\ } \subseteq \widehat{\mathfrak{A}}$ so $\hat{\kappa}_{\mathfrak{A}}(\alpha) = \widehat{\mathfrak{A}} \cap \hat{\kappa}_{C_0(\Phi_{\mathfrak{A}})}(\alpha) = (\operatorname{rng} \nu)\widehat{\ }$. By the semisimplicity of $\mathfrak{A}$, $\kappa_{\mathfrak{A}}(\alpha) = \operatorname{rng} \nu$. ■

1.3. Homomorphisms Between Commutative Group Algebras

In this and the next two sections, we will develop arguments leading to a result analogous to Proposition 1.2.5 for homomorphisms between commutative group algebras. Central to this development are results of P.J. Cohen on certain algebraic properties of the Group and Measure algebras of locally compact Abelian groups. These can be found in the original papers [10, 11], or, perhaps more conveniently, in the third and fourth Chapters of the book [37] of W. Rudin.

1.3.1. Definitions. If Γ is a locally compact Abelian group and E is a closed coset in Γ, the *coset ring* of E, denoted $\mathcal{R}(E)$, is the Boolean ring generated by the relatively open cosets in E. A map $\psi : E_1 \to E_2$ between two cosets is called *affine* if $\psi(\gamma_1 + \gamma_2 - \gamma_3) = \psi(\gamma_1) + \psi(\gamma_2) - \psi(\gamma_3)$ for any $\gamma_1, \gamma_2, \gamma_3 \in E_1$. If $U \subseteq E_1$ and ψ is a map $U \to E_2$ such that there exists a coset E in E_1 with $U \subseteq E$ and an affine map $\psi' : E \to E_2$ extending ψ, then we say that ψ' is an *affine extension* of ψ. A map ψ from $X \subseteq E_1$ into E_2 is called *piecewise affine* if there exists disjoint $S_1, \ldots, S_n \in \mathcal{R}(E_1)$ such that $X = \bigcup_1^n S_k$ and each $\psi|_{S_k}$ has a continuous affine extension. Note that in particular this implies $X \in \mathcal{R}(E_1)$.

We now state the two vital Theorems of P.J. Cohen. Each is stated in terms of group and measure algebras, but it is worth remembering that the Fourier and Fourier-Stieltjes algebras on closed cosets are isomorphic to these algebras, and so these theorems can easily be restated in terms of these algebras.

1.3.2. Theorem. *[10, Theorem 1] If G is a locally compact Abelian group, then a measure $\mu \in M(G)$ is an idempotent if and only if $\hat{\mu} = \chi_S$ for some $S \in \mathcal{R}(\Gamma)$.*

1.3.3. Theorem. *[11, Theorem 1] If $\nu : L^1(G_1) \to M(G_2)$ is a nonzero algebra homomorphism, then there exists a set $Y \in \mathcal{R}(\Gamma_2)$ and a piecewise affine map $\alpha : Y \to \Gamma_1$ such that for each $f \in L^1(G_1)$, $\widehat{\nu(f)} = \hat{f} \circ \alpha$ on Y and $\widehat{\nu(f)} = 0$ off Y. Conversely, any such piecewise affine map determines a homomorphism $\nu : L^1(G_1) \to M(G_2)$, and $\operatorname{rng} \nu \subseteq L^1(G_2)$ if and only if α is proper.*

Before proceeding further, we should note that the apparent conflict in the conclusions of Theorems 1.2.1 and 1.3.3 is easily resolved. If $\nu : L^1(G_1) \to M(G_2)$ is a homomorphism such that $\operatorname{rng} \nu \not\subseteq L^1(G_2)$, then we seem to have on one hand that α is proper, whilst on the other that it is not. The difference in the two theorems is in the domain of α. Indeed, if $\tilde{Y} = \Phi_{M(G_2)} \setminus Z(\operatorname{rng} \nu)$ and $\tilde{\alpha} = \nu^*|_{\tilde{Y}}$ are as in Theorem 1.2.1, and Y, α are as given by Theorem 1.3.3, then $Y = \tilde{Y} \cap \Gamma_2$ and $\alpha = \tilde{\alpha}|_Y$; but since Γ_2 is not necessarily closed in $\Phi_{M(G_2)}$, α need not be proper when $\tilde{\alpha}$ is. Given this, it is not surprising that the two versions of α have different topological properties. This can be seen as an illustration of certain differences between the Gel'fand and Fourier-Stieltjes transforms on measure algebras.

Now that we know the nature of a homomorphism $L^1(G_1) \to L^1(G_2)$, it is opportune to consider the steps required to show that $\kappa(\alpha) = \operatorname{rng} \nu$ for such homomorphisms. By Lemma 1.2.4, what is required is to show that for each $f \in \kappa(\alpha)$, $\hat{f} \circ \alpha^{-1} \in C_0(\alpha(Y))$ has an extension $g \in L^1(G_1)\hat{} = A(\Gamma_1)$. This involves determining the nature of the set $\alpha(Y)$, the behaviour of $\hat{f} \circ \alpha^{-1}$ on $\alpha(Y)$, and the behaviour required of g. For the first two of these, we need to know more about the coset ring and piecewise affine maps. We will devote the next section to this. The behaviour required of g is readily accounted for—much is known of the Fourier algebra on a locally compact Abelian group. One vital property enables us to attain the desired result for certain α at this stage.

1.3.4. Lemma. *If Y is an open coset in Γ_2 and $\alpha : Y \to \Gamma_1$ is affine, proper and continuous, then for any $f \in \kappa(\alpha)$, $\hat{f} \circ \alpha^{-1}$ has an extension in $A(\Gamma_1)$.*

Proof. Since Y is clopen and α is closed and affine, $\alpha(Y)$ is a closed coset. For $\gamma_0 \in Y$, $E = \alpha^{-1}\{\alpha(\gamma_0)\}$ is a compact subcoset of Y, and so $\Lambda = E - \gamma_0$ is a compact subgroup of $Y - \gamma_0$. Clearly $\hat{f} \in A(\Gamma_2)$ is constant on cosets of Λ, so by [37, Theorem 2.7.1], f is concentrated on the open subgroup $H = \mathrm{Ann}_{G_2}(\Lambda)$ of G_2. This means that $f = 0$ off H and $f_1 = f|_H \in L^1(H)$. Now, by [37, Theorem 2.1.2], $\hat{H} \cong \Gamma_2/\Lambda$ via $\langle x, \gamma + \Lambda\rangle = \langle x, \gamma\rangle$ $(x \in H, \gamma + \Lambda \in \Gamma_2/\Lambda)$, and so $\hat{f}_1(\gamma + \Lambda) = \int_H f_1(x)\langle x, \gamma + \Lambda\rangle dx = \int_G f(x)\langle x, \gamma\rangle dx = \hat{f}(\gamma)$. Hence if we let Q_Λ be the quotient map $\Gamma_2 \to \Gamma_2/\Lambda$, then $\hat{f}_1 \circ Q_\Lambda = \hat{f}$. Putting $Y_1 = Q_\Lambda(Y)$, $\hat{f}_1$ is zero off Y_1, a clopen coset in Γ_2/Λ, and $\hat{f}_1|_{Y_1} \in A(Y_1)$.

Now, $\alpha \circ Q_\Lambda^{-1} : Y_1 \to \alpha(Y)$ is a well defined affine bijection. Furthermore, $\alpha \circ Q_\Lambda^{-1}$ is continuous and proper, and hence is an affine homeomorphism onto $\alpha(Y)$. Thus $f \mapsto f \circ \alpha \circ Q_\Lambda^{-1}$ defines an isomorphism from $A(\alpha(Y))$ onto $A(Y_1)$, so $\hat{f} \circ \alpha^{-1} = \hat{f}_1 \circ (\alpha \circ Q_\Lambda^{-1})^{-1} \in A(\alpha(Y))$. Since $\alpha(Y)$ is a closed coset in Γ_2, $A(\alpha(Y)) = A(\Gamma_1)|_{\alpha(Y)}$, so it follows that $\hat{f} \circ \alpha^{-1}$ has an extension in $A(\Gamma_1)$. ∎

It may be of interest at this stage to consider the action of ν in such a case. We will only consider the situation where Y is a subgroup and α is a homomorphism. The case where Y is a coset and α is affine is then formulated by noting that translation in the Fourier algebras $A(\Gamma_1)$ and $A(\Gamma_2)$ corresponds to pointwise multiplication by a character in $L^1(G_1)$ and $L^1(G_2)$.

Let $H_1 = \mathrm{Ann}_{G_1}(\alpha(Y))$, $H_2 = \mathrm{Ann}_{G_2}(Y)$, and let Λ and H be as above. Then $H \subseteq G_2$ and $Y \subseteq \Gamma_2$ are clopen subgroups, $H_2 \subseteq G_2$ and $\Lambda \subseteq \Gamma_2$ are compact subgroups, and $\alpha(Y) \subseteq \Gamma_1$ and H_1 are closed subgroups. We have $\Lambda = \ker \alpha$, so $\psi = \alpha \circ Q_\Lambda^{-1} : Y/\Lambda \to \alpha(Y)$ is a topological group isomorphism whose adjoint $\psi^* : G_1/H_1 \to H/H_2$ is also an isomorphism.

Let $T_{H_1} : L^1(G_1) \to L^1(G_1/H_1)$ be the natural epimorphism, as discussed by Reiter [32, Section 4.3]. Let $\Psi : L^1(G_1/H_1) \to L^1(H/H_2)$ be the isomorphism induced by ψ, and let $I_{H/H_2} : L^1(H/H_2) \to L^1(G_2)$ be the natural monomorphism given by $I_{H/H_2}(f)(x) = f(x + H_2)$ if $x \in H$ and $I_{H/H_2}(f)(x) = 0$ otherwise.

Then for $\iota : \alpha(Y) \hookrightarrow \Gamma_1$ the inclusion mapping, we have $\alpha = \iota \circ \psi \circ Q_\Lambda$, and a corresponding decomposition $\nu = I_{H/H_2} \circ \Psi \circ T_{H_1}$. (Here ι corresponds to T_{H_1}, etc.) Hence if $g \in L^1(G_1)$, $\nu(g)$ is given by

$$x \notin H \implies \nu(g)(x) = 0 \quad \text{a.e.}$$

$$\text{and} \quad x \in H \quad \text{with, say} \quad x' \in G_1 \quad \text{such that} \quad \psi^*(x' + K_1) = x + K_2$$

$$\implies \nu(g)(x) = T_{K_1}(g)(x' + H_1)$$

$$= \int_{K_1} g(x' + t)\, d\lambda_{H_1}(t) \quad \text{a.e.}$$

1.3.5. Corollary. *Suppose Γ_2 is connected, $\nu : L^1(G_1) \to L^1(G_2)$ is a nonzero homomorphism, $Y = \Gamma_2 \setminus Z(\operatorname{rng}\nu)$ and $\alpha = \nu^*|_Y$. Then for some open subgroup H of G_2,*

$$\operatorname{rng}\nu = \kappa(\alpha) = \{f \in L^1(G_2) : f = 0 \text{ off } H\} \cong L^1(H).$$

Proof. Since Γ_2 is connected, $\mathcal{R}(\Gamma_2) = \{\varnothing, \Gamma_2\}$. Let $Y = \Gamma_2 \setminus Z(\operatorname{rng}\nu)$ and $\alpha = \nu^*|_Y$. Then by Theorem 1.3.3, $\varnothing \neq Y \in \mathcal{R}(\Gamma_2)$ and α is a proper piecewise affine map. Thus, by definition, $Y = \Gamma_2$ and α is affine and $Y = \Gamma_2$. Let Λ be the compact subgroup on whose cosets α is contant, and let $H = \mathrm{Ann}_G \Lambda$. Then

$$\begin{aligned} \operatorname{rng} \nu = \kappa(\alpha) &= \left\{ f \in L^1(G) : \hat{f} \text{ is contant on each coset of } \Lambda \right\} \\ &= \left\{ f \in L^1(G) : f = 0 \text{ off } H \right\}, \end{aligned}$$

which is a subalgebra of $L^1(G)$ isomorphic to $L^1(H)$. ∎

In the case where G_2 is also connected, this reduces to $\operatorname{rng} \nu = L^1(G_2)$, and since the Euclidean groups $\mathbb{R}^n$ are the only connected locally compact Abelian groups with connected dual, we have the following.

1.3.6. Corollary. *If $G_2 = \mathbb{R}^n$ for some $n > 0$, then ν is onto.* ∎

1.4. The Coset Ring and Piecewise Affine Maps

As a starting point for considering general piecewise affine maps, we have from the discussion in [37, Section 4.3.4] (cf. [38]), that any set in the coset ring of a locally compact Abelian group Γ is a finite disjoint union of sets in

$$\mathcal{R}_0(\Gamma) = \left\{ E_0 \setminus \left(\bigcup_1^m E_k \right) : m \geq 0, E_0, \ldots, E_m \text{ are open cosets in } \Gamma \right.$$
$$\left. \text{and each of } E_1, \ldots, E_m \text{ are of infinite index in } E_0 \right\}.$$

If E is a closed coset in a locally compact Abelian group, then we can similarly define $\mathcal{R}_0(E)$ and decompose elements of $\mathcal{R}(E)$ into a finite disjoint union of elements of $\mathcal{R}_0(E)$. In what follows, it will be taken as understood that if we introduce a set in the manner "Suppose $S \in \mathcal{R}_0(E)$, say $S = E_0 \setminus \left(\bigcup_1^m E_k\right), \ldots$", then each E_k is an open subcoset in E, and each of $E_1, \ldots, E_m$ are open subcosets of infinite index in E_0.

We also define $\mathcal{R}_d(\Gamma)$, the *discrete coset ring* to be $\mathcal{R}(\Gamma_d)$, being the Boolean ring generated by *all* cosets in Γ. Also define $\mathcal{R}_c(\Gamma) = \{X \in \mathcal{R}_d(\Gamma) : X \text{ is a closed}$

subset of $\Gamma\}$. This is not a Boolean ring (unless Γ is discrete,) but we do have, by [38, Theorem 1.7], that $\mathcal{R}_c(\Gamma) = \{\overline{X} : X \in \mathcal{R}_d(\Gamma)\}$ and also

$$\mathcal{R}_c(\Gamma) = \left\{ \bigcup_1^n X_k : \text{for each } 1 \leq k \leq n \right. \\ \left. \text{there is a closed coset } E_k \subseteq \Gamma \text{ with } X_k \in \mathcal{R}(E_k) \right\}.$$

By applying the above decomposition to each X_k, we can assume, without loss, that each X_k is a closed element of $\mathcal{R}_0(\Gamma_d)$. As discussed in [28], the sets $X \in \mathcal{R}_c(\Gamma)$ are significant when considering Banach-algebraic properties of $L^1(G)$, as the closed ideals of $L^1(G)$ with bounded approximate identity are precisely those with hull in $\mathcal{R}_c(\Gamma)$, and as these hulls are also sets of spectral synthesis, a closed ideal $\mathcal{I}$ of $L^1(G)$ has bounded approximate identity if and only if $\mathcal{I} = \mathcal{I}(X)$ for some $X \in \mathcal{R}_c(\Gamma)$.

It is clear that by [37, Lemma 4.3.3], $\varnothing \notin \mathcal{R}_0(\Gamma)$. Consider $S \in \mathcal{R}_0(\Gamma)$, say $S = E_0 \setminus \left(\bigcup_1^n E_k\right)$, then for all $\gamma \in E_0 - E_0$,

$$S \cap (S+\gamma) = E_0 \setminus \left(\bigcup_1^m E_k \cup \bigcup_1^m (E_k + \gamma) \right) \in \mathcal{R}_0(\Gamma).$$

Hence if E is a coset containing S, then $E \cap (E+\gamma) \neq \varnothing$, so $\gamma \in E - E$. Thus $E_0 - E_0 \subseteq E - E$ and $E_0 \subseteq E$. Hence E_0 is the smallest coset containing S. We call E_0 the coset *generated by* S and denote this $E_0 = E_0(S)$. We now define $N(S)$ to be the minimum number $n \geq 0$ such that there exist open cosets $E_1, \ldots, E_n$ of infinite index in $E_0 = E_0(S)$ such that $S = E_0 \setminus \left(\bigcup_1^n E_k\right)$.

The above decomposition of an element of the coset ring of a locally compact Abelian group can now be applied to obtain a characterization of piecewise affine maps. Let $\psi : X \to \Gamma_1$ be a piecewise affine map as in Definition 1.3.1, then by performing a decomposition as above on each S_k, we can suppose that each S_k is an element of $\mathcal{R}_0(\Gamma_2)$. Also, each $\alpha|_{S_k}$ has a continuous affine extension $\alpha_k : E_k \to \Gamma_1$, for some coset E_k. Then $S_k \subseteq E_k$, so $E_0(S_k) \subseteq E_k$, and $\alpha_k|_{E_0(S_k)}$ is a continuous affine extension of $\alpha|_{S_k}$. Thus we can assume $E_k = E_0(S_k)$. This gives us the following result.

1.4.1. Lemma. *If $X \in \mathcal{R}(\Gamma_2)$ then $\psi : X \to \Gamma_1$ is piecewise affine if and only if there are disjoint $S_1, \ldots, S_n \in \mathcal{R}_0(\Gamma_2)$ such that $X = S_1 \cup \cdots \cup S_n$ and for each k, $\psi|_{S_k}$ has a continuous affine extension $\psi_k : E_0(S_k) \to \Gamma_1$.* ∎

We now present a pair of lemmas that we will use in Section 1.5 to obtain information about the affine maps $\psi_1, \ldots, \psi_n$ from ψ. Each lemma allows us to "smudge" certain $S \in \mathcal{R}(\Gamma)$ to cover a slightly larger set. The first lemma applies to smudge a set $S \in \mathcal{R}_0(\Gamma)$ to cover $E_0(S)$, and the second applies to any $S \in \mathcal{R}(\Gamma)$, to cover $S + \Lambda$, for Λ a compact subgroup of Γ.

1.4.2. Lemma. *Let $S \in \mathcal{R}_0(\Gamma)$ and put $E_0 = E_0(S)$. Then there is a finite subset F of $E_0 - E_0$ such that $E_0 = S + F$.*

Proof. Clearly any $F \subseteq E_0 - E_0$ has $S + F \subseteq E_0$. We prove that for any $S \in \mathcal{R}_0(\Gamma)$ there is a finite set $F \subseteq E_0 - E_0$ with $S + F \supseteq E_0$, by induction on $N(S)$. If $N(S) = 0$, then $S = E_0$, so that $F = \{e\}$ suffices. Now let $n > 0$ and suppose that for any $S' \in \mathcal{R}_0(\Gamma)$ with $N(S') < n$, there is a finite set $F' \subseteq E_0 - E_0$ with $S' + F' = E_0$. Let $S \in \mathcal{R}_0(\Gamma)$ have $N(S) = n$, say $S = E_0 \setminus \left(\bigcup_1^n E_k\right)$. Put $S' = E_0 \setminus \left(\bigcup_1^{n-1} E_k\right)$, then $N(S') \leq n - 1$, so there exists F' with $S' + F' = E_0(S') = E_0$. Then

$$E_0 = S' + F' \subseteq (S \cup E_n) + F' \subseteq (S + F') \cup (E_n + F'),$$

so $E_0 \setminus (E_n + F') \subseteq S + F'$.

Since E_n is of infinite index in E_0, $E_n + F' - F' \subset E_0$, so there exists $\gamma_0 \in E_0 - E_0$ such that $E_n + \gamma_0 \subseteq E_n + F' - F'$. Then $E_n + F'$ and $E_n + (F' + \gamma_0)$ are disjoint. Put $F = F' \cup (F' + \gamma_0)$, then

$$S + F = [S + F'] \cup [S + F' + \gamma_0] \supseteq [E_0 \setminus (E_n + F')] \cup [E_0 \setminus (E_n + (F' + \gamma_0))] = E_0,$$

as required. ∎

1.4.3. Lemma. *If $S \in \mathcal{R}(\Gamma)$ and Λ is a compact subgroup of Γ, then there is a finite subset F of Λ such that $S + \Lambda = S + F$.*

Proof. Let $E_1, \ldots, E_m$ be open cosets such that S is in the Boolean ring generated by $\{E_1, \ldots, E_m\}$. Let $\Xi = \bigcap_1^n (E_k - E_k)$, so that Ξ is an open subgroup of Γ with $S + \Xi = S$. Then $\Lambda \cap \Xi$ is a relatively open subgroup of the compact group Λ, so $\Lambda \cap \Xi$ is of finite index in Λ. Thus there is a finite $F \subseteq \Lambda$ with $(\Lambda \cap \Xi) + F = \Lambda$, and since $S \subseteq S + (\Lambda \cap \Xi) \subseteq S + \Xi = S$, we have $S + F = S + (\Lambda \cap \Xi) + F = S + \Lambda$. ∎

We can now apply these smudging lemmas to an analysis of proper piecewise affine maps.

1.4.4. Lemma. *If $S \in \mathcal{R}_0(G)$ and $\psi : E_0(S) \to \Gamma_1$ is affine such that $\psi|_S$ is proper, then ψ is proper.*

Proof. Applying Lemma 1.4.2, we have that $E_0(S) = \bigcup_1^N (S + \gamma_k - \gamma_0)$, for some $\gamma_0, \gamma_1, \ldots, \gamma_N \in E_0(S)$. If $C \subseteq \Gamma_1$ is compact, each $\gamma \in \psi^{-1}(C)$ lies in some $(S + \gamma_k - \gamma_0)$, so $\gamma - \gamma_k + \gamma_0 \in S$ satisfies $\psi(\gamma - \gamma_k + \gamma_0) \in C - \psi(\gamma_k) + \psi(\gamma_0)$, giving $\gamma \in (\psi|_S)^{-1}(C - \psi(\gamma_k) + \psi(\gamma_0)) + \gamma_k - \gamma_0$. Hence

$$\psi^{-1}(C) \subseteq \bigcup_1^N \Big((\psi|_S)^{-1}\big(C - \psi(\gamma_k) + \psi(\gamma_0)\big) + \gamma_k - \gamma_0 \Big)$$

which is compact, so ψ is proper. ∎

1.4.5. Corollary. *If $S \in \mathcal{R}_0(\Gamma_2)$ and $\psi : S \to \Gamma_1$ is proper with an affine extension ψ', then $\psi'(E_0(S))$ is a closed coset in Γ_1 and $\psi(S) \in \mathcal{R}(\psi'(E_0(S)))$.*

Proof. Without loss, we can assume that ψ' has domain $E_0(S)$. Then, by Lemma 1.4.4, ψ' is proper, and so $E = \psi'(E_0(S))$ is a closed coset in Γ_1. Now, as in Lemma 1.3.4, there is a compact subgroup Λ of $E_0(S) - E_0(S)$ such that $\psi' \circ Q_\Lambda : E_0(S)/\Lambda \to E$ is an affine homeomorphism. Then by Lemma 1.4.3, there is a finite $F \subseteq \Lambda$ with $S + \Lambda = S + F$, giving $S + \Lambda \in \mathcal{R}(E_0(S))$. Hence $Q_\Lambda(S) \in \mathcal{R}(E_0(S)/\Lambda)$ and $\psi(S) = \psi'(S) \in \mathcal{R}(E)$. ∎

If we now apply this result to each piece of a proper piecewise affine map, we obtain a significant property of the range of such a mapping.

1.4.6. Corollary. *If $X \in \mathcal{R}(\Gamma_2)$ and $\psi : X \to \Gamma_1$ is proper and piecewise affine then $\psi(X) \in \mathcal{R}_c(\Gamma_1)$.*

Proof. This follows immediately from Lemma 1.4.1 and the observation that with notation as in Corollary 1.4.5, $\mathcal{R}(E) \subseteq \mathcal{R}_d(\Gamma_1)$. ∎

We now present some results on the coset ring that are easy extensions of [37, Theorem 4.3.3], which states that the union of a finite number of cosets of infinite index in an Abelian group G is a proper subset of G.

1.4.7. Lemma. *Suppose G is an Abelian group and $E_1, \ldots, E_n$ are cosets in G such that $G = \bigcup_1^n E_k$, then G is the union of those of the E_k that are of finite index in G.*

Proof. Let $\mathbb{J} \subseteq \{1, \ldots, n\}$ be those k such that E_k is of finite index in G. For $k \in \mathbb{J}$, let $H_k = E_k - E_k$ a subgroup of finite index in G. Thus $H = \bigcap_{\mathbb{J}} H_k$ is a subgroup of finite index in G. (If $\mathbb{J} = \varnothing$, put $H = G$.) Now, if $x \in G$, then for each $k \notin \mathbb{J}$, $(H + x) \cap E_k$ is empty or a coset of infinite index in H, so by [37, Theorem 4.3.3], $\bigcup_{k \notin \mathbb{J}} (H + x) \cap E_k$ is a proper subset of $H + x$. Thus for some $k \in \mathbb{J}$, $(H + x) \cap E_k \neq \varnothing$, and hence $H + x \subseteq E_k$. Thus $H + x \subseteq \bigcup_{k \in \mathbb{J}} E_k$, and so we have that $G = \bigcup_{k \in \mathbb{J}} E_k$. ∎

1.4.8. Corollary. *Suppose G is an Abelian group and $E_1, \ldots, E_n$ are cosets in G such that $G = \bigcup_1^n E_k$, then for some $1 \le k \le n$, E_k is a subgroup of finite index in G.*

Proof. By Lemma 1.4.7, we may suppose that each E_k has finite index in G. Then e is an element of some E_k, which will be a subgroup. ∎

1.4.9. Corollary. *Suppose Γ_1 and Γ_2 are locally compact Abelian groups and $\alpha : \Gamma_2 \to \Gamma_1$ is a piecewise affine map. Then there is a set $S \in \mathcal{R}_0(\Gamma_2)$ such that $E_0(S)$ is a subgroup of finite index in Γ_2 and $\alpha|_S$ has a continuous affine extension $\alpha_0 : E_0(S) \to \Gamma_1$. Further, if α is proper, then so is α_0.*

Proof. Combining Lemma 1.4.1 with Corollary 1.4.8 gives the existence of S and α_0. The last part follows from Lemma 1.4.4. ∎

It was demonstrated above that the range of a proper piecewise affine map $\alpha : Y \to \Gamma_1$ must be an element of $\mathcal{R}_c(\Gamma_1)$. To obtain a similar result for a general piecewise affine map, we need a generalization of part of Lemma 1.4.3 to show that $S + \Lambda \in \mathcal{R}(\Gamma)$ when $S \in \mathcal{R}(\Gamma)$ and Λ is a closed subgroup of Γ. Such a result is not immediately required for the consideration of homomorphisms between commutative group algebras. It is included here to give a more complete picture of piecewise affine maps, and to provide a basis for some developments in Chapter 3.

First we give some results which could be proven using combinatoric arguments, but yield much more readily to a proof using the connection of the coset ring with the Fourier-Stieltjes algebra.

1.4.10. Lemma. *Suppose Γ is a locally compact Abelian group and $S \in \mathcal{R}_d(\Gamma)$ is a clopen subset of Γ. Then $S \in \mathcal{R}(\Gamma)$ and $\|\chi_S\|_{B(\Gamma)} = \|\chi_S\|_{B(\Gamma_d)}$.*

Proof. By [37, Theorem 1.9.1], $B(\Gamma) = B(\Gamma_d) \cap C(\Gamma)$, with $\|F\|_{B(\Gamma)} = \|F\|_{B(\Gamma_d)}$ for any $F \in B(\Gamma)$, from which the result follows. ∎

1.4.11. Lemma. *If Λ is a closed subgroup of a locally compact Abelian group Γ, and $Q : \Gamma \to \Gamma/\Lambda$ is the quotient mapping, then*

$$\mathcal{R}(\Gamma/\Lambda) = \{Q(S) : S \in \mathcal{R}(\Gamma),\ S + \Lambda = S\},$$

and if $S \in \mathcal{R}(\Gamma)$ has $S + \Lambda = S$, then $\|\chi_S\|_{B(\Gamma)} = \|\chi_{Q(S)}\|_{B(\Gamma/\Lambda)}$.

Proof. Clearly $\mathcal{R}(\Gamma/\Lambda) \subseteq \{Q(S) : S \in \mathcal{R}(\Gamma),\ S + \Lambda = S\}$. Conversely, suppose $S \in \mathcal{R}(\Gamma)$ is such that $S + \Lambda = S$. Let $\mu \in M(G)$ have $\hat{\mu} = \chi_S$, then by [37, Theorem 2.7.1], μ is concentrated on $H = \mathrm{Ann}_G(\Lambda)$. Thus, if μ_H is the restriction of μ to H, given by $\mu_H(E) = \mu(E \cap H)$, then $\mu_H \in M(H)$, and $\hat{\mu}_H \in B(\Gamma/\Lambda)$ is given by $\hat{\mu}_H(\gamma + \Lambda) = \hat{\mu}(\gamma)$. Thus μ_H is an idempotent measure, and $\hat{\mu}_H = \chi_{Q(S)}$, so $Q(S) \in \mathcal{R}(\Gamma/\Lambda)$. Finally, $\|\mu\|_{M(G)} = |\mu|(G) = |\mu|(H) = |\mu_H|(H) = \|\mu_H\|_{M(H)}$, as required. ∎

The following result can be obtained from [38, Theorem 1.3]. We include a proof of it here as a basis for a later result, where we try to estimate the norm of $\chi_{S+\Xi}$ in $B(\Gamma)$.

1.4.12. Proposition. *If $S \in \mathcal{R}(\Gamma)$ and Ξ is a subgroup of Γ, then $S+\Xi \in \mathcal{R}(\Gamma)$.*

Proof. Clearly $S + \Xi$ is clopen, so by Lemma 1.4.10, we only need show that $S+\Xi \in \mathcal{R}_d(\Gamma)$, so we can assume that Γ is discrete. Clearly it suffices to demonstrate the case $S \in \mathcal{R}_0(\Gamma)$, say $S = E_0 \setminus \left(\bigcup_1^n E_k\right)$. For $0 \le k \le n$, put $\Lambda_k = E_k - E_k$. Let $\mathbb{J} \subseteq \{1, \dots, n\}$ be those k such that $\Lambda_k \cap \Xi$ is of finite index in $\Lambda_0 \cap \Xi$, and let $S' = E_0 \setminus \left(\bigcup_{k\in\mathbb{J}} E_k\right)$. Then

$$\gamma \in E_0 \setminus (S + \Xi) \iff \gamma + (\Lambda_0 \cap \Xi) \subseteq \bigcup_1^n E_k$$

$$\iff \Lambda_0 \cap \Xi = \bigcup_1^n (E_k - \gamma) \cap (\Lambda_0 \cap \Xi).$$

Moreover, each $(E_k - \gamma) \cap (\Lambda_0 \cap \Xi)$ is either empty or a translate of $\Lambda_k \cap \Xi$, which will be of finite index in $\Lambda_0 \cap \Xi$ if and only if $k \in \mathbb{J}$. Thus, by Lemma 1.4.7,

$$\bigcup_1^n (E_k - \gamma) \cap (\Lambda_0 \cap \Xi) = \bigcup_{k\in\mathbb{J}} (E_k - \gamma) \cap (\Lambda_0 \cap \Xi),$$

and it follows that $S + (\Lambda_0 \cap \Xi) = S' + (\Lambda_0 \cap \Xi)$.

Put $\Lambda_{\mathbb{J}} = \bigcap_{k\in\mathbb{J}} \Lambda_k$, so that $S' + \Lambda_{\mathbb{J}} \subseteq S'$ and $\Lambda_{\mathbb{J}} \cap \Xi$ is of finite index in $\Lambda_0 \cap \Xi$. Let $F \subseteq \Xi$ be such that $(\Lambda_{\mathbb{J}} \cap \Xi) + F = \Lambda_0 \cap \Xi$, then $S + (\Lambda_0 \cap \Xi) = S' + (\Lambda_0 \cap \Xi) = S' + (\Lambda_{\mathbb{J}} \cap \Xi) + F = S' + F \in \mathcal{R}(\Gamma)$.

Actually, since $S+(\Lambda_0\cap\Xi)\subseteq E_0$, we have $S+(\Lambda_0\cap\Xi)\in\mathcal{R}(E_0)$, and then by Lemma 1.4.11, $Q_{(\Lambda_0\cap\Xi)}(S)\in\mathcal{R}\big(Q_{(\Lambda_0\cap\Xi)}(E_0)\big)$. Now, $\Lambda_0/(\Lambda_0\cap\Xi)\cong(\Lambda_0+\Xi)/\Xi$, via $\gamma+\Lambda_0\cap\Xi\mapsto\gamma+\Xi=Q_\Xi\circ Q^{-1}_{(\Lambda_0\cap\Xi)}(\gamma+\Lambda_0)$, so $Q_\Xi(S)\in\mathcal{R}\big(Q_\Xi(E_0)\big)\subseteq\mathcal{R}(\Gamma/\Xi)$, and thus by Lemma 1.4.11, $S+\Xi\in\mathcal{R}(\Gamma)$. ∎

We then have the following, whose proof is completely analogous to that of Corollary 1.4.6.

1.4.13. Corollary. *If α is a piecewise affine map into a locally compact Abelian group Γ, then* $\operatorname{rng}\alpha\in\mathcal{R}_d(\Gamma)$. ∎

The converse to this is also true—if $X\in\mathcal{R}_d(\Gamma)$, then with $X_d=X\subseteq\Gamma_d$, the identity mapping $X_d\to X$ is a piecewise affine map. (The continuity of this is assured, as X_d has its discrete topology.) We will consider a converse to Corollary 1.4.6 in Section 1.6.

The following result is of no immediate use, apart from demonstrating the utility of Proposition 1.4.12.

1.4.14. Corollary. *If $A,B\in\mathcal{R}(\Gamma)$, then $A+B\in\mathcal{R}(\Gamma)$.*

Proof. We have $A\times B\in\mathcal{R}(\Gamma\times\Gamma)$ and $\{(\gamma,-\gamma):\gamma\in\Gamma\}$ is a closed subgroup of $\Gamma\times\Gamma$. Thus $\big\{(a+\gamma,b-\gamma)\ :\ a\in A,b\in B,\gamma\in\Gamma\big\}\in\mathcal{R}(\Gamma\times\Gamma)$ and $(A+B)\times\{e\}=\big\{(a+\gamma,b-\gamma):a\in A,b\in B,\gamma\in\Gamma\big\}\cap\big(\Gamma\times\{e\}\big)\in\mathcal{R}(\Gamma\times\{e\})$. ∎

1.5. The Range of a Homomorphism Between Commutative Group Algebras

In this section, as promised, we will complete the characterization of the range of a homomorphism between commutative group algebras. We firstly look at the case where $\alpha:Y\to\Gamma_1$ is a piecewise affine map with just one piece, in the manner of Lemma 1.4.1.

1.5.1. Lemma. *Suppose $Y \in \mathcal{R}_0(\Gamma)$ and $\alpha : Y \to \Gamma_1$ is proper with an affine extension $\alpha_1 : E_0(Y) \to \Gamma_1$. Then for any $f \in \kappa(\alpha)$, $\hat{f} \circ \alpha^{-1}$ has an extension in $A(\Gamma_1)$.*

Proof. Put $E = E_0(Y)$, then by Lemma 1.4.4, $\alpha_1 : E \to \Gamma_1$ is proper, so as in Lemma 1.3.4, there is a compact subgroup Λ of $E - E$ such that $\alpha_1 \circ Q_\Lambda^{-1}$ is an affine homeomorphism $E/\Lambda \to \alpha(E)$. Define $\tilde{f} : \Gamma_2 \to \mathbb{C}$ to be the unique function that agrees with $\hat{f}$ on Y, is constant on cosets of Λ, and is zero off $Y + \Lambda$. That is,

$$\tilde{f}(\gamma) = \begin{cases} f(\gamma - \lambda) & \text{when } \lambda \in \Lambda \text{ is such that } \gamma - \lambda \in Y; \\ 0 & \text{if } \gamma \notin Y + \Lambda. \end{cases}$$

To show $\tilde{f} \in A(\Gamma_2)$, we require sets $S_1, \ldots, S_n \in \mathcal{R}(\Gamma_2)$ such that $\bigcup_1^n S_j = Y + \Lambda$ and $\tilde{f} \cdot \chi_{S_1}, \ldots, \tilde{f} \cdot \chi_{S_n} \in A(\Gamma_2)$, for then

$$\tilde{f} = \tilde{f} \cdot \chi_{S_1 \cup \cdots \cup S_n} = \sum_{k=1}^{n} \tilde{f} \cdot \chi_{S_k} \cdot \chi_{\Gamma \setminus (S_1 \cup \cdots \cup S_{k-1})} \in A(\Gamma_2).$$

By Lemma 1.4.3, there is a finite set $F \subseteq \Lambda$ such that $Y + \Lambda = \bigcup_{\lambda \in F}(Y + \lambda)$. Then for each $\lambda \in F$, $\tilde{f} \cdot \chi_{Y+\lambda} = f \circ \tau_{-\lambda} \in A(\Gamma_2)$, so the sets $\{Y + \lambda\}_{\lambda \in F} \subseteq \mathcal{R}(\Gamma_2)$ are as required, giving $\tilde{f} \in A(\Gamma_2)$. Furthermore, the defined properties of $\tilde{f}$ mean that $\tilde{f} \in \hat{\kappa}(\alpha_1)$. Hence by Lemma 1.3.4, $\tilde{f} \circ \alpha_1^{-1}$ has an extension in $A(\Gamma_1)$, and since $\tilde{f} \circ \alpha_1^{-1}$ is itself an extension of $\hat{f} \circ \alpha^{-1}$, we are done. ■

The final proof of the general case requires a result on ideals of commutative group algebras, so that the individual extensions obtained by Lemma 1.5.1 can be combined.

1.5.2. Lemma. *If $\mathcal{I}$ and $\mathcal{J}$ are closed ideals of a Banach algebra $\mathfrak{A}$ and $\mathcal{I}$ contains $\{e_n\}_{n \in \Delta}$, a bounded approximate right identity for $\mathcal{I}$, then $\mathcal{I} + \mathcal{J}$ is a closed ideal of $\mathfrak{A}$.*

Proof. Clearly $\mathcal{I} + \mathcal{J}$ is an ideal of $\mathfrak{A}$. Let $\pi : \mathcal{J}/(\mathcal{I} \cap \mathcal{J}) \to (\mathcal{I} + \mathcal{J})/\mathcal{I}$ be the natural isomorphism. Clearly π is continuous. Let $y \in \mathcal{J}$ be such that $\|y + \mathcal{I}\| < 1$, so that there exists $x \in \mathcal{I}$ with $\|y - x\| < 1$. Then

$$\begin{aligned}\|\pi^{-1}(y+\mathcal{I})\| &= \inf_{z\in\mathcal{I}\cap\mathcal{J}} \|y+z\| \leq \inf_{n\in\Delta} \|y - ye_n\| \\ &\leq \inf_{n\in\Delta} \left(\|y-x\| + \|x - xe_n\| + \|e_n\|\|x-y\|\right) \\ &< 1 + 0 + \sup_{n\in\Delta}\|e_n\|\end{aligned}$$

and so π^{-1} is continuous. Hence $(\mathcal{I}+\mathcal{J})/\mathcal{I}$ is complete, and it follows that $\mathcal{I}+\mathcal{J}$ is a closed ideal of $\mathfrak{A}$. ∎

1.5.3. Lemma. *If $A, B \in \mathcal{R}_c(\Gamma)$ then $\mathcal{I}(A) + \mathcal{I}(B) = \mathcal{I}(A \cap B)$.*

Proof. By [28, Theorem 13], $\mathcal{I}(A)$ has a bounded approximate identity. Hence, by Lemma 1.5.2, we have that $\mathcal{I}(A) + \mathcal{I}(B)$ is a closed ideal of $A(\Gamma)$. Furthermore, $Z(\mathcal{I}(A) + \mathcal{I}(B)) = A \cap B \in \mathcal{R}_c(\Gamma)$. By [15, Theorem 1], $A \cap B$ is a set of synthesis, so $\mathcal{I}(A) + \mathcal{I}(B) = \mathcal{I}(A \cap B)$. ∎

1.5.4. Corollary. *Suppose $A, B \in \mathcal{R}_c(\Gamma)$, $f, g \in L^1(G)$ and $\hat{f}|_{A\cap B} = \hat{g}|_{A\cap B}$. Then there exists $h \in L^1(G)$ such that $\hat{f}|_A = \hat{h}|_A$ and $\hat{g}|_B = \hat{h}|_B$.*

Proof. We have $f - g \in \mathcal{I}(A\cap B) = \mathcal{I}(A) + \mathcal{I}(B)$, so there are $f' \in \mathcal{I}(A)$ and $g' \in \mathcal{I}(B)$ such that $f - g = -f' + g'$. Then $h = f + f' = g + g' \in L^1(G)$ has $\hat{f}|_A = \hat{h}|_A$ and $\hat{g}|_B = \hat{h}|_B$. ∎

1.5.5. Theorem. *If $Y \in \mathcal{R}(\Gamma_2)$ and $\alpha : Y \to \Gamma_1$ is a proper piecewise affine map, then for any $f \in \kappa(\alpha)$, $\hat{f} \circ \alpha^{-1}$ has an extension in $A(\Gamma_1)$.*

Proof. By Lemma 1.4.1, there exist disjoint $S_1, \ldots, S_n \in \mathcal{R}_0(\Gamma_2)$ such that $\bigcup_1^n S_k = Y$ and each $\alpha|_{S_k}$ is proper with an affine extension $\alpha_k : E_0(S_k) \to \Gamma_1$. For each $1 \leq k \leq n$, $\hat{f} \cdot \chi_{S_k} \in \hat{\kappa}(\alpha|_{S_k})$, so by Lemma 1.5.1, there exists $g_k \in A(\Gamma_1)$ such that $g_k|_{\alpha(S_k)} = \hat{f} \circ (\alpha|_{S_k})^{-1} = \hat{f} \circ \alpha^{-1}|_{\alpha(S_k)}$.

By Corollary 1.4.6, each $\alpha(S_k) \in \mathcal{R}_c(\Gamma_1)$, and so we can apply Corollary 1.5.4 repeatedly to show that $\hat{f} \circ \alpha^{-1}$ has an extension in $A(\Gamma_1)$. ∎

We now have the main theorem of this section.

1.5.6. Theorem. *Suppose G_1 and G_2 are locally compact Abelian groups and $\nu : L^1(G_1) \to L^1(G_2)$ is an algebra homomorphism. Then with $Y = \Gamma_2 \setminus Z(\operatorname{rng}\nu)$ and $\alpha = \nu^*|_Y$, $\operatorname{rng}\nu = \kappa(\alpha)$.*

Proof. Combine Lemma 1.2.4 with Theorem 1.5.5. ■

Once we have this theorem, the following generalization requires only a small amount of extra effort.

1.5.7. Theorem. *If $\nu : L^1(G_1) \to L^1(G_2)$ is an algebra homomorphism and $\mathcal{J}$ is a closed ideal in $L^1(G_1)$, then $\nu(\mathcal{J})$ is a closed subalgebra of $L^1(G_2)$.*

Proof. Let $\mathcal{I} = \ker\nu = \mathcal{I}(\alpha(Y))$, where Y and α are as above. By Corollary 1.4.6, $\alpha(Y) \in \mathcal{R}_c(\Gamma_1)$. Hence, by [28, Theorem 13], $\mathcal{I}$ has a bounded approximate identity. Then, by Lemma 1.5.2, $\mathcal{I}+\mathcal{J}$ is a closed ideal of $L^1(G_1)$. Since ν is maps onto $\kappa(\alpha)$, a Banach space, we have by the Open Mapping Theorem that $\nu(L^1(G_1) \setminus (\mathcal{I}+\mathcal{J}))$ is open in $\kappa(\alpha)$. Hence $\nu(\mathcal{J}) = \kappa(\alpha) \setminus \nu(L^1(G_1) \setminus (\mathcal{I}+\mathcal{J}))$ is closed in $\kappa(\alpha)$, and an ideal of $\kappa(\alpha)$, so that $\nu(\mathcal{J})$ is a closed subalgebra of $L^1(G_2)$. ■

Given the rather definite characterization of the range of homomorphisms between commutative group algebras, it is natural to look for generalizations and similar results for homomorphisms between other Banach algebras. Some of these generalizations will be dealt with later, in Section 1.6 and Chapter 4—we comment here on possible generalizations of Theorem 1.5.6 to a situation involving general locally compact groups. (That is, those that are not necessarily Abelian.)

There are two possible generalization of Theorem 1.5.6 in this direction—the first is to consider homomorphisms between the group algebras of locally compact groups, and the second is to consider homomorphisms between the Fourier algebras of locally compact groups. In the latter case, the actual algebras are commutative semisimple Banach algebras, and so there is the possibility of using Theorem 1.2.1

and attaining a result analogous to Theorem 1.5.6 in this case. One impediment to this is the lack of a complete result characterizing homomorphisms $A(G_1) \to A(G_2)$ similar to Theorem 1.3.3. Some partial results have been attained by B. Host in [22]. These require that G_1 have an Abelian subgroup of finite index. Other difficulties arise in trying to generalize Lemma 1.5.3. There has been some investigation of the existence of bounded approximate identities in closed ideals of the Fourier algebra of a locally compact group by B. Forrest in [14]. These seem to require that the group, G_1 in this case, be amenable.

The second possible generalization is that of homomorphisms between the group algebras of locally compact groups. Here we no longer have commutative Banach algebras, so Theorem 1.2.1 does not apply. One could, however, develop analogous ideas involving the structure spaces of these algebras, and look to characterize the range this way. A more modest goal, perhaps, would be to show that the range of such a homomorphism is closed, or that if a homomorphism is dense-ranged, then it is onto. Some detail of the possible homomorphisms would assist in this task, but unfortunately there are only partial results in this direction. For example, see [25].

Note that by Proposition 1.1.2, we can characterize the range of group algebra homomorphisms $L^1(G_1) \to L^1(G_2)$ in the case where G_2 is Abelian.

1.6. Piecewise Affine Sets

In this section, we will place all the results considered in the previous three sections into a much more general, and possibly more natural, setting. We have, for a coset E, definitions of $A(E)$, $B(E)$, $\mathcal{R}(E)$, and so on, derived from those for a locally compact Abelian group. We consider a more abstract type of set X for which we can define similar objects. Most of the proofs in this section can be deduced from those for the group and measure algebras on a locally compact group, and so the proofs herein will be indications as to how this can be done.

As a starting point, we have, by Corollary 1.4.13 and the subsequent discussion, that a subset of a locally compact Abelian group Γ is the range of a piecewise affine map if and only if it is in the discrete coset ring of Γ. For *proper* piecewise affine maps, we have by Corollary 1.4.6 that if $X \subseteq \Gamma$ is the range of a proper piecewise affine map, then $X \in \mathcal{R}_c(\Gamma)$. The converse to this is not immediately forthcoming.

Note that, by the Structure Theorem for locally compact Abelian groups, any locally compact Abelian group G is topologically isomorphic to $\mathbb{R}^n \times G_0$, where G_0 contains a compact open subgroup, and so G_0 does not contain any subgroup topologically isomorphic to $\mathbb{R}$. Thus n is the maximum non-negative integer such that G contains a subgroup topologically isomorphic to $\mathbb{R}^n$. We generalize this to subsets of a locally compact Abelian group.

1.6.1. Definitions. Define a *Euclidean* coset $E \subseteq \Gamma$ to be one that is affinely homeomorphic to $\mathbb{R}^N$, for some $N \geq 0$. In such a case, N is the *dimension* of E. A *Euclidean subcoset* of a set $S \subseteq \Gamma$ is a subset E of S that is a Euclidean coset in Γ. The *Euclidean dimension*, $\dim_{\mathbb{R}} S$, of a set $S \subseteq \Gamma$ is the largest $n \geq 0$ such that there is a Euclidean subcoset of S of dimension n. A set $S \subseteq \Gamma$ is *uniformly Euclidean* if each maximal Euclidean subcoset E has the same Euclidean dimension as S.

Suppose $X_1, \ldots, X_m \in \mathcal{R}_c(\Gamma)$, each X_k has Euclidean dimension $\leq n$, and $X = \bigcup_1^m X_k$. Then for any Euclidean coset $E \subseteq X$, $E = \bigcup_1^m (E \cap X_k)$, and so one of the sets $E \cap X_k$ must have an interior point in E. However, $E \cap X_k \in \mathcal{R}_c(E)$, and since any proper subcoset of E has empty interior, it follows from [38, Theorem 1.7] (as in the introduction to Section 1.4), that there is some $S_k \subseteq E \cap X_k$ with $S_k \in \mathcal{R}(E) = \{\varnothing, E\}$. Hence $E \subseteq X_k$, and so X has Euclidean dimension $\leq n$. Hence if each X_k is uniformly Euclidean of dimension n, then X is also uniformly Euclidean of dimension n.

1.6.2. Proposition. *Suppose Γ_1 is a locally compact Abelian group and $X \subseteq \Gamma_1$. Then X is the range of a proper piecewise affine map if and only if $X \in \mathcal{R}_c(\Gamma_1)$ and X is uniformly Euclidean.*

Proof. We first show that the range of a proper piecewise affine map is uniformly Euclidean. It suffices, by Lemma 1.4.1 and the discussion above, to show that if Γ_2 is a locally compact Abelian group, $S \in \mathcal{R}_0(\Gamma_2)$, and $\alpha : E_0(S) \to \Gamma_1$ is a proper affine map, then $\alpha(S)$ is uniformly Euclidean, with $\dim_{\mathbb{R}} \alpha(S) = \dim_{\mathbb{R}} \Gamma_2$.

By the Structure Theorem for locally compact Abelian groups, [37, Theorem 2.4.1], we can assume that $\Gamma_2 = \Gamma_2' \times \mathbb{R}^N$, where Γ_2' has a compact open subgroup. Then $S \subseteq \mathcal{R}_0(\Gamma_2)$ has $S = S' \times \mathbb{R}^N$, for some $S' \in \mathcal{R}_0(\Gamma_2')$. Also, if $\gamma \in S$, then $\alpha^{-1}\{\alpha(\gamma)\}$ is a compact coset in Γ_2, so that $\Lambda = \alpha^{-1}\{\alpha(\gamma)\} - \gamma$ is a compact subgroup of Γ_2. Clearly $\Lambda = \Lambda' \times \{0\}$ for some compact subgroup Λ' of Γ_2'.

Since $Q_\Lambda(S) = Q_{\Lambda'}(S') \times \mathbb{R}^N$, and $Q_{\Lambda'}(S')$ has no nontrivial Euclidean subcosets, it follows that $Q_\Lambda(S)$ is uniformly Euclidean of dimension N. Moreover, $\alpha \circ Q_\Lambda^{-1} : Q_\Lambda(E_0) \to \alpha(E_0)$ is an affine homeomorphism, so that $\alpha(S)$ is also uniformly Euclidean of dimension N.

Conversely, suppose X is uniformly Euclidean of dimension N. We have that $X = S_1 \cup \cdots \cup S_n$, where each S_k is contained within a closed coset E_k in Γ_1 such that $S_k \in \mathcal{R}(E_k)$. Then for each k, $\Gamma_k = E_k - E_k$ has a closed subgroup Γ_k' such that $\Gamma_k \cong \Gamma_k' \times \mathbb{R}^N$ and Γ_k' has a compact (relatively) open subgroup Ξ_k. Let $\psi_k : \Gamma_k' \times \mathbb{R}^N \to E_k$ be an affine homeomorphism and let $S_k' \in \mathcal{R}(\Gamma_k')$ be such that $\psi_k(S_k' \times \mathbb{R}^N) = S_k$. Now define Γ', Y and α as follows:

$$\Gamma' = \mathbb{R}^N \times \prod_{k=1}^{n} \Gamma_k' \times \mathbb{Z},$$

$$Y_k = \mathbb{R}^N \times \Xi_1 \times \cdots \times \Xi_{k-1} \times S_k' \times \Xi_{k+1} \times \cdots \times \Xi_n \times \{k\} \in \mathcal{R}(\Gamma'),$$

$$Y = \bigcup_{k=1}^{n} Y_k$$

$$\text{and} \quad \gamma' = (x_1, \ldots, x_N, \xi_1, \ldots, \xi_{k-1}, \gamma, \xi_{k+1}, \ldots, \xi_n, k) \in Y$$

$$\implies \alpha(\gamma') = \psi_k(\gamma, x_1, \ldots, x_N).$$

Clearly $\alpha : Y \to \Gamma$ is a piecewise affine map with $\alpha(Y) = X$. Moreover, since each piece $\alpha|_{Y_k}$ is essentially a quotient by the compact subgroup $\prod_{j \neq k} \Xi_j$, each $\alpha|_{Y_k}$ is proper, and so α is proper. ■

Thus $(\mathbb{R} \times \{0\}) \cup (\mathbb{Z} \times \{1\}) \subseteq \mathbb{R}^2$ is not the range of a proper continuous piecewise affine map, despite being an element of $\mathcal{R}_c(\mathbb{R}^2)$. Similar sets are easily constructed in any locally compact Abelian group of nonzero Euclidean dimension.

1.6.3. Corollary. *A closed ideal $\mathcal{I} \subseteq L^1(G)$ is the kernel of a homomorphism into a commutative group algebra if and only if $Z(\mathcal{I})$ is a uniformly Euclidean element of $\mathcal{R}_c(\Gamma)$.*

Proof. If $\mathcal{I}$ is the kernel of $\nu : L^1(G) \to L^1(G')$, then by Proposition 1.2.1, $Z(\mathcal{I}) = \alpha(Y)$, so that $Z(\mathcal{I})$ is a uniformly Euclidean element of $\mathcal{R}_c(\Gamma)$. Conversely, if $\mathcal{I}$ is a closed ideal such that $X = Z(\mathcal{I})$ is a uniformly Euclidean element of $\mathcal{R}_c(\Gamma)$, then by Proposition 1.6.2, there is a locally compact Abelian group Γ', a set $Y \in \mathcal{R}(\Gamma)$, and a proper piecewise affine map $\alpha : Y \to \Gamma$ with range X. Then with $\nu : L^1(G) \to L^1(G')$ the homomorphism determined by α, we have $Z(\ker \nu) = \alpha(Y) = X = Z(\mathcal{I})$. Since X is a set of spectral synthesis, $\ker \nu = \mathcal{I}$. ∎

If we now examine the proof of Theorem 1.5.5, no use is made of the fact that $\alpha(Y)$, the set on which $\hat{f} \circ \alpha^{-1}$ is defined, is uniformly Euclidean. Thus we have a result slightly stronger than Theorem 1.5.5. This stronger result can be stated as considering proper piecewise affine maps $\alpha : Y \to \Gamma$ where Y is a set made up of pieces whose Euclidean dimension is not the same. The definition we make has a little in common with that of a manifold, with some significant differences. The intent is the same—we seek a definition of a topological space with associated structures that give it certain properties similar to those of a locally compact Abelian group, whilst discarding aspects of a locally compact Abelian group that are irrelevant for our considerations.

1.6.4. Definitions. A *piecewise affine set* is a locally compact topological space X with a finite *atlas* of associated *charts* $\psi_k : S_k \to X$ $(1 \le k \le n)$ satisfying :

(i) each S_k is in the coset ring of a locally compact Abelian group Γ_k, (note: $\Gamma_1, \Gamma_2, \ldots$ may be distinct)

(ii) each ψ_k is a homeomorphism onto X_k, its range,

(iii) $\bigcup_1^n X_k = X$

(iv) if $X_j \cap X_k$ is nonempty then it is equal to some X_i,

(v) if $X_j \subseteq X_k$ then $\psi_k^{-1} \circ \psi_j : S_j \to \Gamma_k$ is a proper piecewise affine injection. (Equivalently, a piecewise affine map that is a homeomorphism onto its range.)

If X is a piecewise affine set with respect to two different atlases $\{\psi_k\}_1^n$ and $\{\psi'_j\}_1^m$, we say that these atlases are *compatible* if there is an atlas with $\{\psi_k\}_1^n \cup \{\psi'_j\}_1^m$ as a subset. We can now define the *coset ring*, *Fourier algebra*, and *Fourier-Stieltjes algebra* of such a set X by

$$\mathcal{R}(X) = \{S \subseteq X : \psi_k^{-1}(S) \in \mathcal{R}(\Gamma_k)\ (1 \le k \le n)\}$$

$$A(X) = \{f \in C_0(X) : f \circ \psi_k = f_k|_{S_k}, \text{ for some } f_k \in A(\Gamma_k)\ (1 \le k \le n)\}$$

$$B(X) = \{F \in C(X) : F \circ \psi_k = F_k|_{S_k}, \text{ for some } F_k \in B(\Gamma_k)\ (1 \le k \le n)\}.$$

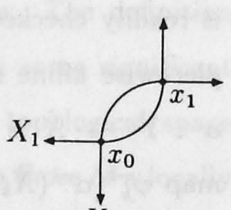
Suppose $\mathbb{I} \subseteq \{1, \ldots, n\}$ is such that $X = \bigcup_{\mathbb{I}} X_k$. Then it is clear that $\mathcal{R}(X) = \{S \subseteq X : \psi_k^{-1}(S) \in \mathcal{R}(\Gamma_k)\ (k \in \mathbb{I})\}$. Similarly, we only need consider $k \in \mathbb{I}$ when assessing whether a given function is an element of $A(X)$ or $B(X)$. For this reason, compatible atlases give identical definitions of $\mathcal{R}(X)$, $A(X)$ and $B(X)$. We define $\mathcal{R}_d(X)$ and $\mathcal{R}_c(X)$ analogously to the group case. We make $A(X)$ into a Banach algebra by defining $\|f\| = \sum_1^n \|f_k \chi_{S_k}\|_{A(\Gamma_k)}$, and similarly for $B(X)$. It is readily checked that compatible atlases give equivalent norms. If $\tilde{X}$ is another piecewise affine set, (with charts $\tilde{\psi}_j : \tilde{S}_j \to \tilde{X}_j$, etc,) and $Y \in \mathcal{R}(X)$, a map $\alpha : Y \to \tilde{X}$ is called *piecewise affine* if each $\tilde{\psi}_j^{-1} \circ \alpha \circ \psi_k$ is a piecewise affine map $\psi_k^{-1}(\alpha^{-1}(\tilde{X}_k)) \to \tilde{\Gamma}_j$. We will also speak of this as a *morphism* of piecewise affine sets, and adopt the usual gamut of terms *monomorphism*, *epimorphism*, *isomorphism*, *topological isomorphism*, *automorphism*, ... Note that these all include the continuity criterion that originally occurred in 1.3.1, where we defined piecewise affine.

Note. When it is intended that the term "piecewise affine map" should refer to the definition given above, this intention will be stated, if it is not clear from the context. Otherwise, we will mean a piecewise affine map between locally compact Abelian groups. Thus, if we say that a set is the range of a piecewise affine map, then we mean a piecewise affine map as in definition 1.3.1, rather than a map of the type in the above definition.

1.6.5. Example. Suppose $\Gamma_1, \ldots, \Gamma_n$ are locally compact Abelian groups and for each k, let $X_k \in \mathcal{R}(\Gamma_k)$, and let $X = \dot{\bigcup}_1^n X_k$. Then X is clearly a piecewise affine set.

1.6.6. Example. Let Γ be a locally compact Abelian group and let $X \in \mathcal{R}_c(\Gamma)$. We have, by [38, Theorem 1.7], that $X = \bigcup_1^n X_k$, where each X_k is in the coset ring of E_k, a closed coset in Γ. For each nonempty $F \subseteq \{1, \ldots, n\}$ such that $\bigcap_{k \in F} X_k \neq \varnothing$, put $X_F = \bigcap_{k \in F} X_k$. With $\gamma_F \in X_F$, $\Gamma_F = \bigcap_{k \in F} E_k - \gamma_F$ is a closed subgroup of Γ and $X_F - \gamma_F \in \mathcal{R}(\Gamma_F)$. Let $S_F = X_F - \gamma_F$, and let $\psi_F : S_F \to X$ be the translation $\gamma \mapsto \gamma + \gamma_F$. Then (i)-(iv) above are clear. If $X_F \subseteq X_{F'}$, then Γ_F is a closed subgroup of $\Gamma_{F'}$ and $\psi_{F'}^{-1} \circ \psi_F : S_F \to \Gamma_{F'}$ has an affine extension $\Gamma_F \to \Gamma_{F'}$ given by $\gamma \mapsto \gamma + \gamma_F - \gamma_{F'}$, which is proper. Thus we have (v), so that any $X \in \mathcal{R}_c(\Gamma)$ can be viewed as a piecewise affine set in a natural way. Due to the comments within definition 1.6.4, we can usually ignore all the X_F except $X_1, \ldots, X_n$.

1.6.7. Example. Let $X = \bigcup_1^3 X_n$, where $S_1 = S_2 = \mathbb{R}$, $S_3 = \{0,1\} \in \mathcal{R}(\mathbb{Z})$, $X_1 \cap X_2 = X_3 = \{x_0, x_1\}$, and the charts are given by $\psi_1(0) = \psi_2(0) = \psi_3(0) = x_0$ and $\psi_1(1) = \psi_2(1) = \psi_3(1) = x_1$.



1.6.8. Definitions. We say that a piecewise affine set X is *disjoint* if there is a topological isomorphism from X onto a set of the type in Example 1.6.5. We say that a piecewise affine set X can be embedded in a group (or just *embedded*) if there is a topological isomorphism from X onto a set of the type in Example 1.6.6. (Or equivalently, if there is a proper monomorphism from X into a locally compact

Abelian group.) A disjoint piecewise affine set can be embedded, whereas the set in Example 1.6.7 cannot be embedded. We will chiefly be interested in those piecewise affine sets that can be embedded.

The following theorem contains a representative sample of some of the possible results on piecewise affine sets.

1.6.9. Theorem. *Suppose X and $\widetilde{X}$ are piecewise affine sets, $W \in \mathcal{R}(X)$ and $\alpha : W \to \widetilde{X}$ is piecewise affine. Then*

(i) $\Phi_{A(X)} = X$, via $\varphi_x(\gamma) = \gamma(x)$, and $A(X)$ is semisimple and regular,

(ii) $A(X)$ is a closed ideal of $B(X)$,

(iii) $B(X)$ is the multiplier algebra of $A(X)$,

(iv) the idempotents in $B(X)$ are $\{\chi_Y : Y \in \mathcal{R}(X)\}$,

(v) any $X \in \mathcal{R}_c(X)$ is a set of synthesis for $A(X)$,

(vi) a closed ideal $\mathcal{I}$ of $A(X)$ has bounded approximate identity if and only if $Z(\mathcal{I}) \in \mathcal{R}_c(X)$,

(vii) $\alpha(W) \in \mathcal{R}_d(\widetilde{X})$, and if α is proper, then $\alpha(W) \in \mathcal{R}_c(\widetilde{X})$,

(viii) $f \mapsto f \circ \alpha$ defines an algebra homomorphism $A(\widetilde{X}) \to B(X)$, which has range in $A(X)$ if and only if α is proper, in which case the range is $\kappa_{A(X)}(\alpha)$.

Further, if $\widetilde{X}$ can be embedded, and $\nu : A(\widetilde{X}) \to A(X)$ is an algebra homomorphism, then

(ix) $Y = X \setminus Z(\operatorname{rng} \nu) \in \mathcal{R}(X)$ and $\nu^*|_Y$ is piecewise affine.

Proof. Most of (i)...(vi), and the first part of (viii), can be deduced from the fact that we have the natural algebra epimorphisms

$$\rho_{X_k} : B(X) \to B(X_k) \cong B(S_k) \cong \mathcal{I}_{B(\Gamma_k)}(\Gamma_k \setminus S_k) \subseteq B(\Gamma_k),$$

for which $\quad \rho_{X_k}(A(X)) \subseteq A(X_k) \cong A(S_k) \cong \mathcal{I}_{A(\Gamma_k)}(\Gamma_k \setminus S_k) \subseteq A(\Gamma_k),$

and that $\rho_{X_1} \oplus \cdots \oplus \rho_{X_n} : A(X) \to \bigoplus_1^n A(X_k)$ is a monomorphism. The statement (vii) follows directly from the definition of piecewise affine sets and maps, and

Corollaries 1.4.6 and 1.4.13. Once we have the form of $\alpha(W)$ for proper α, the proof that the range of the homomorphism $f \mapsto f \circ \alpha$ is $\kappa(\alpha)$ involves showing that $A(\widetilde{X})|_{\alpha(W)} = A\big(\alpha(W)\big)$, which we can prove as in 1.5.6. To show (ix), note that we have (without loss) $\widetilde{X} \in \mathcal{R}_c(\widetilde{\Gamma})$, so $A(\widetilde{X}) \cong A(\widetilde{\Gamma})/\mathcal{I}(\widetilde{X})$. Thus we have a homomorphism $\rho_{\widetilde{X}} : A(\widetilde{\Gamma}) \to A(\widetilde{X})$, from which we obtain homomorphisms $\rho_{X_k} \circ \nu \circ \rho_{\widetilde{X}} : A(\widetilde{\Gamma}) \to A(\Gamma_k)$, which we can then tackle using Theorem 1.3.3. ∎

A useful application of this is a description of $L^1(G)/\mathcal{I}$, for certain ideals $\mathcal{I}$. If $\mathcal{I} = \mathcal{I}(X)$ for some $X \in \mathcal{R}_c(\Gamma)$, then the inclusion mapping $\iota_X : X \hookrightarrow \Gamma$ is a proper piecewise affine injection, so that $f \mapsto f \circ \iota_X = f|_X$ defines the homomorphism $\rho_X : A(\Gamma) \to A(X)$ whose range is $\kappa_{A(X)}(\iota_X) = A(X)$. Thus ρ_X is an epimorphism with kernel $\mathcal{I}(X)$, so that $L^1(G)/\mathcal{I}(X) \cong A(X)$.

Using this, along with a construction similar to Proposition 1.6.2, and parts of Theorem 1.6.9, it is possible to obtain some characterizations of those ideals with hull in $\mathcal{R}_c(\Gamma)$. It is interesting to compare this with [28, Theorem 13], which characterizes exactly the same ideals as is done here.

1.6.10. Proposition. *If G is a locally compact Abelian group and $\mathcal{I}$ is a closed ideal of $L^1(G)$, then the following are equivalent:*

(i) $Z(\mathcal{I}) \in \mathcal{R}_c(\Gamma)$,

(ii) there exist locally compact Abelian groups $\Gamma_1, \dots, \Gamma_n$, $Y \in \mathcal{R}\big(\dot\bigcup \Gamma_k\big)$ *and a proper piecewise affine map* $\alpha : Y \to \Gamma$ *with range* $Z(\mathcal{I})$,

(iii) $\mathcal{I}$ *is the kernel of a homomorphism* $L^1(G) \to L^1(G_1) \oplus \cdots \oplus L^1(G_n)$,

(iv) $L^1(G)/\mathcal{I}$ *is isomorphic to a closed subalgebra of a finite direct sum of commutative group algebras, and*

(v) $L^1(G)/\mathcal{I}$ *is isomorphic to the Fourier algebra of an embeddable piecewise affine set.*

1.7. Subalgebras of Commutative Group Algebras

In this section we will examine closed subalgebras of commutative group algebras, which we call *group subalgebras*, and for certain classes of these, develop necessary and sufficient conditions for property **(G)**. We assume throughout that G is a locally compact Abelian group with dual Γ.

The following is a direct consequence of Proposition 1.1.2 and Theorem 1.5.6.

1.7.1. Proposition. *A closed subalgebra $\mathfrak{A}$ of $L^1(G)$ has property* **(G)** *if and only if $Y = \Gamma \setminus Z(\mathfrak{A}) \in \mathcal{R}(\Gamma)$ and there is a locally compact Abelian group Γ' and a proper piecewise affine map $\alpha : Y \to \Gamma'$ with $\mathfrak{A} = \kappa(\alpha)$.* ■

We now consider specific classes of group subalgebras and develop necessary and sufficient conditions for amenability and property **(G)**. The simplest such class is that of the closed ideals of commutative group algebras.

1.7.2. Theorem. *Let $\mathcal{I}$ be a closed ideal of $L^1(G)$, and put $E = Z(\mathcal{I})$. Then $\mathcal{I}$ is amenable if and only if $E \in \mathcal{R}_c(\Gamma)$, whereas $\mathcal{I}$ has property* **(G)** *if and only if $E \in \mathcal{R}(\Gamma)$. In either case, $\mathcal{I} = \mathcal{I}(E)$.*

Proof. The first part of this is [29, Theorem 1]. This relies on the fact that a closed ideal of an amenable Banach algebra is amenable if and only if it has bounded approximate identity. For the second, we have by Proposition 1.7.1 that if $\mathcal{I}$ is an ideal with property **(G)**, then $Y = \Gamma \setminus E \in \mathcal{R}(\Gamma)$, and so $E \in \mathcal{R}(\Gamma)$. Conversely, if $E = Z(\mathcal{I}) \in \mathcal{R}(\Gamma)$, then E is clopen, and consequently of synthesis, so that $\mathcal{I} = \mathcal{I}(E)$. Moreover, if we define $\alpha : Y \to \Gamma$ to be the inclusion mapping, then α is a proper piecewise affine map with $\kappa(\alpha) = \mathcal{I}(E)$, so by the above discussion, $\mathcal{I}$ has property **(G)**. ■

Remark. In the above proof, the epimorphism $\nu : L^1(G) \to \mathcal{I}$ determined by α has $\widehat{\nu(f)} = \chi_{\Gamma\setminus E} \cdot \hat{f}$. Then by [10, Theorem 1], there is an idempotent measure $\mu \in M(G)$ with $\hat{\mu} = \chi_{\Gamma\setminus E}$, so that ν is given by $f \mapsto f * \mu$. So we can see that

ν is a multiplicative projection. The problem of finding a projection onto an ideal of a group algebra is quite a different problem. Some results on this problem are contained in Appendix B.

We now turn to another construction of closed subalgebras of $L^1(G)$ that are amenable and yet lack property (G). Suppose $\mathfrak{A}$ is a commutative Banach algebra and H is a finite group of automorphisms of $\mathfrak{A}$. Put

$$\mathfrak{A}_H = \{a \in \mathfrak{A} : h(a) = a,\ (h \in H)\}$$

then $\mathfrak{A}_H$ is a closed subalgebra of $\mathfrak{A}$. (Note that here, we write H as a multiplicative group, as it is not necessarily Abelian.) We then have the following result, whose proof in this generality was kindly suggested by Professor B.E. Johnson.

1.7.3. Proposition. *If $\mathfrak{A}$ is a commutative amenable Banach algebra and H is a finite group of automorphisms of $\mathfrak{A}$, then $\mathfrak{A}_H$ is an amenable Banach algebra.*

Proof. Let $\{d_n\}_{n\in\Delta} \subseteq \mathfrak{A} \hat{\otimes} \mathfrak{A}$ be an approximate diagonal for $\mathfrak{A}$, and let H have identity ι and cardinality N. Put $K = \max_{h\in H}\|h\|$.

The group $H \times H$ can be made into a group of automorphisms on $\mathfrak{A} \hat{\otimes} \mathfrak{A}$ via $(h_1, h_2)(a_1 \otimes a_2) = h_1(a_1) \otimes h_2(a_2)$ and $\mathfrak{A}_H \hat{\otimes} \mathfrak{A}_H = (\mathfrak{A} \hat{\otimes} \mathfrak{A})_{(H\times H)}$. For each $n \in \Delta$, put $d'_n = \frac{1}{N}\sum_{h\in H}(h, h)(d_n)$. Then $\{d'_n\}_{n\in\Delta}$ is an approximate diagonal for $\mathfrak{A}$ with $(h, h)(d'_n) = d'_n$, for each $h \in H$; let $M = \sup_{n\in\Delta}\|d'_n\|$. Now put

$$d''_n = e \otimes e - \prod_{h\in H}\big(e \otimes e - (h, \iota)(d'_n)\big),$$

where this product is in the algebra $\mathfrak{A} \hat{\otimes} \mathfrak{A}$, and the term $e \otimes e$ plays a purely formal rôle as a multiplicative identity. It is clear that $\{d''_n\}_{n\in\Delta}$ is a bounded net in $\mathfrak{A} \hat{\otimes} \mathfrak{A}$. Moreover, if $(h_1, h_2) \in H \times H$, then

$$\begin{aligned}(h_1, h_2)(d''_n) &= e \otimes e - \prod_{h\in H}\big(e \otimes e - (h_1 h, h_2)(d'_n)\big)\\ &= e \otimes e - \prod_{h\in H}\big(e \otimes e - (h_1 h h_2^{-1}, \iota)(d'_n)\big) = d''_n,\end{aligned}$$

so that $d''_n \in \mathfrak{A}_H \hat{\otimes} \mathfrak{A}_H$. Also, if $a \in \mathfrak{A}_H$ then

$$\begin{aligned} \left\| a - a\pi(d''_n) \right\| &= \left\| a \prod_{h \in H} \pi\big(e \otimes e - (h, \iota)(d'_n)\big) \right\| \\ &\leq \left\| a - a\pi(d'_n) \right\| \prod_{h \in H \setminus \{\iota\}} \left\| e \otimes e - (h, \iota)(d'_n) \right\| \\ &\leq \left\| a - a\pi(d'_n) \right\| (1 + KM)^{N-1} \\ &\to 0, \end{aligned}$$

so that $\{\pi(d''_n)\}_{n \in \Delta}$ is an approximate right identity for $\mathfrak{A}_H$. Finally, we have

$$d''_n = \sum_{\varnothing \neq S \subseteq H} (-1)^{|S|} \prod_{h \in S} (h, \iota)(d'_n),$$

so if we let $S \mapsto h_S \in S$ be a choice function, then for each $a \in \mathfrak{A}_H$,

$$\begin{aligned} \left\| d''_n \cdot a - a \cdot d''_n \right\| &\leq \sum_{\varnothing \neq S \subseteq H} \left\| (h_S, \iota) d'_n \cdot a - a \cdot (h_S, \iota) d'_n \right\| \prod_{h \in S \setminus \{h_S\}} \left\| (h, \iota) \right\| \left\| d'_n \right\| \\ &\leq \sum_{\varnothing \neq S \subseteq H} \left\| (h_S, \iota) \right\| \left\| d'_n \cdot a - h_S^{-1}(a) \cdot d'_n \right\| (KM)^{|S|-1} \\ &\leq (2^N - 1) K \left\| d'_n \cdot a - a \cdot d'_n \right\| (KM)^{N-1} \\ &\to 0. \end{aligned}$$

Hence $\{d''_n\}_{n \in \Delta}$ is an approximate diagonal for $\mathfrak{A}_H$, and so $\mathfrak{A}_H$ is amenable. ∎

So we see that if G is a locally compact Abelian group and H is a finite group of automorphisms of $\mathfrak{A} = L^1(G)$, then $\mathfrak{A}_H$ is amenable. We denote this algebra $L^1_H(G)$. To determine when $L^1_H(G)$ has property (G), note that, by [11, Theorem 1], the automorphisms of $L^1(G)$ are characterized by the piecewise affine homeomorphisms of Γ. Hence we can consider H as a finite group of piecewise affine homeomorphisms $\Gamma \to \Gamma$. Then

$$\begin{aligned} L^1_H(G) &= \left\{ f \in L^1(G) : \hat{f} \circ h = \hat{f} \ (h \in H) \right\} \\ &= \left\{ f \in L^1(G) : \hat{f} \text{ is constant on each orbit } H(\gamma) \right\} \end{aligned}$$

So, applying Proposition 1.7.1, we see that $L^1_H(G)$ has property (G) if and only if there is a proper piecewise affine map α from $Y = \Gamma \setminus Z(L^1_H(G))$ into some other locally compact Abelian group such that $L^1_H(G) = \kappa(\alpha) = \{f \in L^1(G) : \hat{f}(\Gamma \backslash Y) = 0$ and $\hat{f}$ is constant on each $\alpha^{-1}\{\alpha(\gamma)\}\}$. So it would seem that the partition of Γ into orbits $H(\gamma)$ is identical to the partition of Y into sets on which α is constant. The following lemma delivers precisely this result.

1.7.4. Lemma. *Suppose $\nu : L^1(G') \to L^1(G)$ is a homomorphism between commutative group algebras, $Y = \Gamma \setminus Z(\operatorname{rng} \nu)$ and $\alpha = \nu^*|_Y$, and suppose H is a finite group of piecewise affine homeomorphisms of Γ. If $\operatorname{rng} \nu = L^1_H(G)$, then $Y = \Gamma$ and for $\gamma_1, \gamma_2 \in \Gamma$, $\alpha(\gamma_1) = \alpha(\gamma_2) \iff H(\gamma_1) = H(\gamma_2)$.*

Proof. For each $\gamma \in \Gamma$, $H(\gamma)$ is finite, and since $L^1(G)\hat{}$ separates points of Γ, there exists $f \in L^1(G)$ with $\hat{f}(H(\gamma)) = \{1\}$. Put $\tilde{f} = \frac{1}{|H|}\sum_h \hat{f} \circ h$; then $\tilde{f} \in (L^1_H(G))\hat{}$ and $\tilde{f}(\gamma) = 1$. Hence $\gamma \in \Gamma \setminus Z(L^1_H(G)) = Y$, so $Y = \Gamma$.

Now suppose $\gamma_1, \gamma_2 \in \Gamma$ are such that $H(\gamma_1) = H(\gamma_2)$. For each $f \in L^1(G')$, $\nu(f) \in L^1_H(G)$. Hence $\widehat{\nu(f)}(\gamma_1) = \widehat{\nu(f)}(\gamma_2)$, so that $\hat{f}(\alpha(\gamma_1)) = \hat{f}(\alpha(\gamma_2))$, and since $A(\Gamma')$ separates points, $\alpha(\gamma_1) = \alpha(\gamma_2)$.

On the other hand, if $H(\gamma_1) \neq H(\gamma_2)$, then $H(\gamma_1)$ and $H(\gamma_2)$ are finite disjoint sets, so there exists $f \in L^1(G)$ with $\hat{f}(H(\gamma_1)) = \{0\}$ and $\hat{f}(H(\gamma_2)) = \{1\}$. Thus $\tilde{f} = \frac{1}{|H|}\sum_h \hat{f} \circ h \in (L^1_H(G))\hat{} = (\kappa(\alpha))\hat{}$ and $\tilde{f}(\gamma_1) \neq \tilde{f}(\gamma_2)$, so $\alpha(\gamma_1) \neq \alpha(\gamma_2)$. ∎

We now use the above to show that if H is a finite group of automorphisms of Γ, then $L^1_H(G)$ rarely has property **(G)**. This is a natural situation to consider, for then we can consider H as a group of automorphisms on G, being the group of adjoints of elements of H. (This reverses the multiplication on H, which is immaterial in the current situation.) Then $L^1_H(G) = \{ f \in L^1(G) : f \circ h = f \ (h \in H) \}$, which is $L^1(G^H)$, a convolution algebra on the *orbit hypergroup* $G^H = \{ H(g) : g \in G \}$. See [39] for more information on the amenability of hypergroups and hypergroup algebras.

In the situation where $\kappa(\alpha) = L^1_H(G)$, we have seen that we have $Y = \Gamma$. The following lemmas allow us to obtain further special properties of such a piecewise affine map.

1.7.5. Lemma. *Suppose $S \in \mathcal{R}_0(\Gamma)$ is such that $E_0(S)$ is a subgroup of finite index in Γ, and H is a finite group of automorphisms of Γ. Put $\tilde{S} = \bigcap_{h\in H} h(S)$, then $\tilde{S} \in \mathcal{R}_0(\Gamma)$ and $E_0(\tilde{S}) = \bigcap_{h\in H} h(E_0(S))$ is a subgroup of finite index in Γ.*

Proof. Suppose $S = E_0 \setminus (\bigcup_1^m E_k)$, as in the definition of $\mathcal{R}_0(\Gamma)$, and put $\tilde{E}_0 = \bigcap_{h\in H} h(E_0)$. Each of $\{h(E_0) : h \in H\}$ is a subgroup of finite index in Γ, so $\tilde{E}_0$ is a subgroup of finite index in Γ. Also, $\tilde{S} = \tilde{E}_0 \setminus (\bigcup_{h\in H} \bigcup_1^m (h(E_k) \cap \tilde{E}_0))$, with each $h(E_k) \cap \tilde{E}_0$ being empty or of infinite index in $\tilde{E}_0$. Hence $\tilde{S} \in \mathcal{R}_0(\Gamma)$ and $E_0(\tilde{S}) = \tilde{E}_0$. ■

In the following, we assume the notation of Lemma 1.7.4, and apply Lemma 1.7.5 and Corollary 1.4.9.

1.7.6. Corollary. *If $\operatorname{rng} \nu = L^1_H(G)$, then there exists $S \in \mathcal{R}_0(\Gamma)$ such that $E_0(S)$ is a subgroup of finite index in Γ, $\alpha|_S$ has a proper continuous affine extension $E_0(S) \to \Gamma'$, and both S and $E_0(S)$ are H-invariant.* ■

The following theorem characterizes property (**G**) in the algebras $L^1_H(G)$ that we have been considering. Part of the proof is based on some of the ideas in Section 1.6, but only as far as considering piecewise affine sets that are disjoint unions of locally compact Abelian groups.

1.7.7. Theorem. *Suppose H is a finite group of automorphisms of a locally compact Abelian group Γ. Then the following are equivalent :*

*(i) $L^1_H(G)$ has property (**G**),*

(ii) the subgroup $\Lambda = \{\gamma \in \Gamma : H(\gamma) = \{\gamma\}\}$ is of finite index in Γ, and

(iii) $L^1_H(G)$ is isomorphic to a finite direct sum of group algebras.

Proof. Supposing (i), then by Proposition 1.7.1 and Lemma 1.7.4, there is a locally compact Abelian group G' and a proper piecewise affine map $\alpha : \Gamma \to \Gamma'$ such that the level sets of α are precisely the orbits of the action of H on Γ. By Corollary 1.7.6, there exists $S \in \mathcal{R}_0(\Gamma)$ such that $E_0(S)$ is a subgroup of finite index in Γ, $\alpha|_S$ has a proper continuous affine extension $\alpha_0 : E_0(S) \to \Gamma'$ and for each $h \in H$, $h(S) = S$ and $h\big(E_0(S)\big) = E_0(S)$.

Now, for each $h \in H$ and each $\gamma \in S$, $\alpha_0\big(h(\gamma)\big) = \alpha\big(h(\gamma)\big) = \alpha(\gamma) = \alpha_0(\gamma)$, so that $\Lambda_0 = \big\{\gamma \in E_0(S) : \alpha_0 \circ h(\gamma) = \alpha_0(\gamma),\ (h \in H)\big\}$ is a subgroup of $E_0(S)$ with $S \subseteq \Lambda_0$. Since $E_0(S)$ is the coset generated by S, we have that $\Lambda_0 = E_0(S)$, and so $\alpha_0 \circ h = \alpha_0$, $(h \in H)$. Put $\Xi = \{\gamma \in E_0(S) : \alpha_0(\gamma) = \alpha_0(e)\} = \alpha_0^{-1}\{\alpha_0(e)\}$, a subgroup of $E_0(S)$. For each $\gamma \in S$, $H(\gamma) \subseteq S$, so

$$\begin{aligned} \gamma' \in H(\gamma) &\iff \alpha(\gamma') = \alpha(\gamma) \\ &\iff \alpha_0(\gamma') = \alpha_0(\gamma) \text{ and } \gamma' \in S, \end{aligned}$$

so $H(\gamma) = (\gamma + \Xi) \cap S$. Thus $\{\gamma \in E_0(S) : H(\gamma) \subseteq \gamma + \Xi\}$, a subgroup of $E_0(S)$, contains S. It follows that $H(\gamma) \subseteq \gamma + \Xi$ for all $\gamma \in E_0(S)$. For each $h \in H$, let $\tilde{h} : E_0(S) \to \Xi$ be the homomorphism defined by $\tilde{h}(\gamma) = h(\gamma) - \gamma$, so that $\Lambda \cap E_0(S) = \bigcap_{h \in H} \tilde{h}^{-1}\{e\}$. It remains to be proven that Ξ is finite, for then each $\tilde{h}^{-1}\{e\}$ is of finite index in $E_0(S)$, which is in turn of finite index in Γ.

By Lemma 1.4.2, and the fact that $E_0(S)$ is a subgroup of Γ there exists $\gamma_1, \ldots, \gamma_N \in E_0(S)$ such that $E_0(S) = \bigcup_1^N \gamma_k + S$, giving

$$\Xi = \Xi \cap E_0(S) = \bigcup_{1 \le k \le N} \gamma_k + \big((\Xi - \gamma_k) \cap S\big) = \bigcup_{1 \le k \le N} \gamma_k + H(-\gamma_k),$$

which is evidently finite.

Now assume (ii). For each coset $\gamma + \Lambda$ of Λ, and each $h \in H$, $h(\gamma + \Lambda)$ is the coset $h(\gamma) + \Lambda$, so that H acts on Γ/Λ. Let $H(\gamma_1 + \Lambda), \ldots, H(\gamma_N + \Lambda)$ be the orbits of this action, and for each $1 \le k \le N$, let $h_{k1}, \ldots, h_{kn_k} \in H$ be such that the cosets of Λ that make up $H(\gamma_k + \Lambda)$ are $\big\{h_{kj}(\gamma_k + \Lambda) : 1 \le j \le n_k\big\}$.

For each $1 \le k \le N$, $H_k = \big\{h \in H : h(\gamma_k) \in \gamma_k + \Lambda\big\}$ is a subgroup of H, and $\Lambda_k = \big\{h(\gamma_k) - \gamma_k : h \in H_k\big\}$ is a subgroup of Λ. Furthermore, H_k acts on $\gamma_k + \Lambda$ by $h(\gamma_k + \lambda) = (\gamma_k + \lambda) + \big(h(\gamma_k) - \gamma_k\big)$, that is, by translations by elements of Λ_k. For $1 \le j \le n_k$, define $\alpha_{kj} : h_{kj}(\gamma_k) + \Lambda \to \Lambda/\Lambda_k$ by $\alpha_{kj}\big(h_{kj}(\gamma_k) + \lambda\big) = \lambda + \Lambda_k$.

This is continuous and affine, and since $\alpha_{kj}^{-1}(\lambda + \Lambda_k) = h_{kj}(\gamma_k) + \lambda + \Lambda_k$ is finite, α_{kj} is also proper.

Each coset of Λ in Γ is of the form $h_{kj}(\gamma_k) + \Lambda$, for some unique k and j, so we can define a proper piecewise affine map $\alpha : \Gamma \to \Lambda/\Lambda_1 \cup \cdots \cup \Lambda/\Lambda_N$ by "piecing together" all the α_{kj}. For each $\gamma \in \Gamma$, say $\gamma = h_{kj}(\gamma_k) + \lambda$, we have $H(\gamma) = H\big(h_{kj}(\gamma_k + \lambda)\big) = H(\gamma_k + \lambda)$. Also $\alpha_{kj}^{-1}(\lambda + \Lambda_k) = h_{kj}(\gamma_k + \lambda + \Lambda_k) = h_{kj}\big(H_k(\gamma_k + \lambda)\big)$. Hence

$$\alpha^{-1}\{\alpha(\gamma)\} = \bigcup_{1\le j\le n_k} \alpha_{kj}^{-1}(\lambda + \Lambda_k) = \bigcup_{1\le j\le n_k} h_{kj}\big(H_k(\gamma_k + \lambda)\big) = H(\gamma_k + \lambda) = H(\gamma),$$

and as this holds for each $\gamma \in \Gamma$, $\kappa(\alpha) = L^1_H(G)$. Now, by Theorem 1.6.9, α determines a homomorphism $\nu : A(\Lambda/\Lambda_1) \oplus \cdots \oplus A(\Lambda/\Lambda_N) \to A(\Gamma)$ with range $\kappa(\alpha)$. Also, $\ker \nu = \mathcal{I}\big(\mathrm{rng}(\alpha)\big)$ and since α is surjective, we have that ν is a monomorphism. Hence $A_H(\Gamma) = \kappa(\alpha) \cong A(\Lambda/\Lambda_1) \oplus \cdots \oplus A(\Lambda/\Lambda_N)$.

The last implication (iii) $\implies$ (i) follows from Proposition 0.2.5. ∎

So we see that the amenable algebras of the form $L^1_H(G)$ will usually fail to have property **(G)**. For instance, if Γ is connected, then for $L^1_H(G)$ to have property **(G)**, we must have $\Lambda = \Gamma$, and so $H = \{\iota\}$ and $L^1_H(G) = L^1(G)$.

If G is a locally compact Abelian group, we always have the automorphism η on G given by $x \mapsto -x$. (Although occasionally we have $\eta = \iota$, as we will see.) Then $H = \{\iota, \eta\}$ is a finite group of automorphisms of G and $L^1_H(G) = L^1_{\mathrm{sym}}(G)$, the subalgebra of symmetric (or even) functions in $L^1(G)$. We now apply the preceding theorem to this case.

1.7.8. Theorem. *If G is a locally compact Abelian group, the following are equivalent :*

(i′) $L^1_{\mathrm{sym}}(G)$ *has property* **(G)**,

(ii′) $G \cong \sum_{\mathfrak{a}} \mathbb{Z}_2 \times \prod_{\mathfrak{b}} \mathbb{Z}_2 \times F$, *for some cardinals* $\mathfrak{a}$ *and* $\mathfrak{b}$ *and some finite group* F, *and*

(iii′) *$L^1_{\mathrm{sym}}(G)$ is isomorphic to a group algebra.*

Proof. Suppose (i′), then by Theorem 1.7.7, $\Lambda = \{\gamma \in \Gamma : H(\gamma) = \{\gamma\}\}$ is of finite index in Γ, say $|\Gamma/\Lambda| = N$. Then for each $\gamma \in \Gamma$, $N \cdot \gamma \in \Lambda$ and so $2N \cdot \gamma = e$. Hence Γ is of bounded order. By [20, Theorem A.25], there is an *algebraic* isomorphism $\psi : \Gamma_d \to \sum_{i\in\mathbb{I}} \mathbb{Z}_{n_i}$, where $\mathbb{I}$ is an index set and $\{n_i : i \in \mathbb{I}\}$ is a bounded set of integers greater than 2. But $\{\gamma \in \Gamma : 2 \cdot \gamma = e\} = \Lambda$ is of finite index, so $F = \psi^{-1}(\sum_{n_i>2} \mathbb{Z}_{n_i})$ is a finite subgroup of Γ with $(\Gamma/F)_d \cong \sum_{n_i=2} \mathbb{Z}_2$.

Let Λ_0 be a compact open subgroup of Γ, which we can assume to contain F. If we now apply the argument of the above paragraph to $\hat{\Lambda}_0$, we obtain that $\Lambda_0 \cong F \times \prod_{\mathfrak{a}} \mathbb{Z}_2$, for some cardinal $\mathfrak{a}$. By continuing with an argument similar to that used in [20, 25.29], or by a straightforward application of Zorn's Lemma, we can obtain a complement to Λ_0, which will be isomorphic to $\sum_{\mathfrak{b}} \mathbb{Z}_2$ for some cardinal $\mathfrak{b}$, giving $G \cong F \times \sum_{\mathfrak{a}} \mathbb{Z}_2 \times \prod_{\mathfrak{b}} \mathbb{Z}_2$.

For the implication (ii′) $\Longrightarrow$ (iii′), we show that for $G = \sum_{\mathfrak{a}} \mathbb{Z}_2 \times \prod_{\mathfrak{b}} \mathbb{Z}_2 \times F$, $L^1_{\mathrm{sym}}(G)$ is isomorphic to a group algebra. Let $H = \sum_{\mathfrak{a}} \mathbb{Z}_2 \times \prod_{\mathfrak{b}} \mathbb{Z}_2$ so that $G = H \times F$ and $H^{(2)} = \{e\}$. With $\Psi : L^1(G) \to L^1(H) \hat{\otimes} \ell^1(F)$ the natural isomorphism, it is easily verified that $\Psi(L^1_{\mathrm{sym}}(G)) = L^1(H) \hat{\otimes} \ell^1_{\mathrm{sym}}(F)$. Now, $\ell^1_{\mathrm{sym}}(F)$ is a finite-dimensional commutative semisimple algebra over $\mathbb{C}$, so $\ell^1_{\mathrm{sym}}(F) \cong \mathbb{C}^m \cong \ell^1(\mathbb{Z}_m)$, where $m = \dim(\ell^1_{\mathrm{sym}}(F))$. Thus $L^1_{\mathrm{sym}}(G) \cong L^1(H) \hat{\otimes} \ell^1(\mathbb{Z}_m) \cong L^1(H \times \mathbb{Z}_m)$.

The final implication (iii′) $\Longrightarrow$ (i′) is trivial. ∎

In light of the conclusion (iii′) in Theorem 1.7.8, it is natural to ask whether we can reach the same conclusion for the algebras considered in Theorem 1.7.7. We will give an example of an Abelian group G with a finite group of automorphisms H such that $\ell^1_H(G)$ has property (G), but is not isomorphic to a group algebra.

1.7.9. Example. Let U and V be as constructed in [26, p.616-7]. That is, U is a countably infinite torsion-free Abelian group and V is a non-isomorphic subgroup that is of index 2 in U. Let $\Upsilon = \hat{U}$, a connected compact Abelian group. Then

$\Xi = \mathrm{Ann}_\Upsilon(V)$ is a two-element group, say $\Xi = \{e, \xi\}$, and $\Upsilon/\Xi = \hat{V}$ is also compact and connected. Put $G = U \times \mathbb{Z}_2$, so that $\Gamma = \Upsilon \times \mathbb{Z}_2$, and define $\eta \in \mathrm{Aut}(\Gamma)$ by $\eta(v, 0) = (v, 0)$ and $\eta(v, 1) = (v + \xi, 1)$. Then $\eta^2 = \iota$, so $H = \{\iota, \eta\}$ is a finite group of automorphisms of Γ, and $\{\gamma \in \Gamma : H(\gamma) = \{\gamma\}\} = \Upsilon \times \{0\}$, which is of finite index in Γ, so by Theorem 1.7.7, $\ell^1_H(G)$ has property (G). Thus $\ell^1_H(G)$ is isomorphic to a finite direct sum of group algebras. In fact, if we apply the construction in the proof of Theorem 1.7.7, we obtain $\ell^1_H(G) \cong \ell^1(U) \oplus \ell^1(V)$, which has carrier space $\Upsilon \cup \Upsilon/\Xi$.

Suppose $\ell^1(U) \oplus \ell^1(V)$ is isomorphic to a group algebra $L^1(G')$, so that there exists a piecewise affine homeomorphism $\alpha : \Upsilon \cup \Upsilon/\Xi \to \Gamma'$. Thus Γ' has two connected components, which are necessarily affinely homeomorphic. It follows that Υ and Υ/Ξ are topologically isomorphic, and so U and V are isomorphic. (Contradiction.)

Many thanks to Dr Laci Kovács and his group U for so perfectly meeting the specifications desired.

(iii') $L^1_H(G)$ is isomorphic to a group algebra.

Proof. Suppose (i), then by Theorem 1.7.7, $\Lambda = \{\gamma \in \Gamma : H(\gamma) = \{\gamma\}\}$ is of finite index in Γ, say $|\Gamma/\Lambda| = N$. Then for each $\gamma \in \Gamma$, $N \cdot \gamma \in \Lambda$ and so $2N \cdot \gamma = e$. Hence Γ is of bounded order. By [20, Theorem A.25], there is an algebraic isomorphism $\varphi : \Gamma_d \to \sum_{\iota \in I} \mathbb{Z}_{n_\iota}$, where I is an index set and $\{n_\iota : \iota \in I\}$ is a bounded set of integers greater than 2. But $\{\gamma \in \Gamma : 2 \cdot \gamma = e\} = \Lambda$ is of finite index, so $F = \varphi^{-1}(\sum_{\iota \in I, n_\iota > 2} \mathbb{Z}_{n_\iota})$ is a finite subgroup of Γ with $(\Gamma/F)_d \cong \sum \mathbb{Z}_2$.

Let Λ_0 be a compact open subgroup of Γ, which we can assume to contain [illegible] the above paragraph to Λ_0, we obtain that [illegible] $\sum \mathbb{Z}_2$ for some cardinal a. By combining with an argument similar to [illegible] application of Zorn's Lemma, we [illegible] to $\sum \mathbb{Z}_2$ for some cardinal b, giving $G \cong F \times \sum_a \mathbb{Z}_2 \times \prod_b \mathbb{Z}_2$.

For the implication (ii') $\Longrightarrow$ (iii') we need that for $G = \sum_a \mathbb{Z}_2 \times \prod_b \mathbb{Z}_2 \times F$, $L^1_{sym}(G)$ is isomorphic to a group algebra. Let $H = \sum_a \mathbb{Z}_2 \times \prod_b \mathbb{Z}_2$ so that $G = H \times F$ and $H^{(2)} = \{e\}$. With $\Phi : L^1(G) \to L^1(H) \hat{\otimes} \ell^1(F)$ the natural isomorphism, it is easily verified that $\Phi(L^1_{sym}(G)) = L^1(H) \hat{\otimes} \ell^1_{sym}(F)$. Now, $\ell^1_{sym}(F)$ is a finite-dimensional commutative semisimple algebra over $\mathbb{C}$, so $\ell^1_{sym}(F) \cong \mathbb{C}^m \cong \ell^1(\mathbb{Z}_m)$, where $m = \dim(\ell^1_{sym}(F))$. Then $L^1_{sym}(G) \cong L^1(H) \hat{\otimes} \ell^1(\mathbb{Z}_m) \cong L^1(H \times \mathbb{Z}_m)$.

The final implication (iii') $\Longrightarrow$ (i) is trivial. ∎

In light of the conclusion (iii') in Theorem 1.7.8, it is natural to ask whether we can reach the same conclusion for the algebras considered in Theorem 1.7.7. We will give an example of an Abelian group G with a finite group of automorphisms H such that $L^1_H(G)$ has property (G), but is not isomorphic to a group algebra.

1.7.9. Example. Let U and V be as constructed in [26, p.616-7]. That is, U is a countably infinite torsion-free Abelian group and V is a non-isomorphic subgroup that is of index 2 in U. Let $\mathrm{T} = \hat{U}$, a connected compact Abelian group. Then

Chapter 2. Property (G) in Unital Banach Algebras

In this chapter we examine some amenable Banach algebras which we show to lack property (G) by methods entirely different to those in Chapter 1. For the results presented herein, I am indebted to the suggestion concerning the Cuntz algebras and their relevant properties by U. Haagerup, and its communication through P.C. Curtis Jr. and G.A. Willis.

2.1. Extensions of Homomorphisms

Suppose ν is a dense-ranged homomorphism from a group algebra $L^1(G)$ into a unital Banach algebra $\mathfrak{A}$. Then $\mathfrak{A}^{-1}$, the set of invertible elements of $\mathfrak{A}$, is open in $\mathfrak{A}$, so that there is some $f \in L^1(G)$ with $\nu(f)$ invertible. We start with a result dealing with a situation more general that this.

2.1.1. Lemma. *Suppose $\mathfrak{A}$ and $\mathfrak{B}$ are unital Banach algebras and $\mathcal{I}$ is a closed left ideal of $\mathfrak{A}$ with a left approximate identity $\{e_n\}_{n\in\Delta}$. If $\nu : \mathcal{I} \to \mathfrak{B}$ is a continuous homomorphism with $\operatorname{rng}\nu \cap \mathfrak{B}^{-1} \neq \varnothing$, then there is a unique homomorphism $\tilde{\nu} : \mathfrak{A} \to \mathfrak{B}$ extending ν. Moreover, $\tilde{\nu}(e) = e = \lim_{n\in\Delta} \nu(e_n)$, $\overline{\nu(\mathcal{I})} = \overline{\tilde{\nu}(\mathfrak{A})}$, and if $\{e_n\}_{n\in\Delta}$ is bounded in norm by $M > 0$, then $\|\tilde{\nu}\| \leq M\|\nu\|$.*

Proof. Suppose $a \in \mathcal{I}$ is such that $\nu(a) \in \mathfrak{B}^{-1}$. Then any homomorphism $\tilde{\nu} : \mathfrak{A} \to \mathfrak{B}$ extending ν must satisfy $\tilde{\nu}(x) = \nu(xa)\big[\nu(a)\big]^{-1}$, $(x \in \mathfrak{A})$. Define $\tilde{\nu}$ to be exactly this. Then $\tilde{\nu}$ is a continuous linear extension of ν with $\tilde{\nu}(e) = e$.

For each $n \in \Delta$, and each $x \in \mathfrak{A}$,

$$\nu(xe_n) - \tilde{\nu}(x) = \nu\big(x(e_n a - a)\big)\big[\nu(a)\big]^{-1} \to 0$$

so $\tilde{\nu}(x) = \lim_{n\in\Delta} \nu(xe_n)$. It is now clear that $\tilde{\nu}(\mathfrak{A}) \subseteq \overline{\nu(\mathcal{I})}$, $e = \tilde{\nu}(e) = \lim_{n\in\Delta} \nu(e_n)$, and that if each $\|e_n\| \leq M$, then $\|\tilde{\nu}\| \leq M\|\nu\|$. It remains to be shown that $\tilde{\nu}$ is multiplicative. If $x, y \in \mathfrak{A}$, then $ya \in \mathcal{I}$, so $\nu(xya) = \lim_{n\in\Delta} \nu(xe_n ya)$. However, $\nu(xya) = \tilde{\nu}(xy)\nu(a)$ and $\lim_{n\in\Delta} \nu(xe_n ya) = [\lim_{n\in\Delta} \nu(xe_n)]\nu(ya) = \tilde{\nu}(x)\tilde{\nu}(y)\nu(a)$, and since $\nu(a) \in \mathfrak{A}^{-1}$, we have that $\tilde{\nu}(xy) = \tilde{\nu}(x)\tilde{\nu}(y)$, as desired. ∎

We now apply the above to the situation where $L^1(G)$ is a closed ideal, with bounded approximate identity, of $M(G)$.

2.1.2. Proposition. *Suppose $\mathfrak{A}$ is a Banach algebra with unit e, G is a locally compact group, and $\nu : L^1(G) \to \mathfrak{A}$ is a continuous homomorphism such that $\operatorname{rng}\nu \cap \mathfrak{A}^{-1} \neq \varnothing$. Then ν has a unique extension to a homomorphism $\tilde{\nu} : M(G) \to \mathfrak{A}$. Further, $\|\tilde{\nu}\| = \|\nu\|$ and $\overline{\nu(L^1(G))} = \overline{\tilde{\nu}(\ell^1(G))} = \overline{\tilde{\nu}(M(G))}$.*

Proof. Let Δ be the set of compact neighbourhoods of $e \in G$, ordered by $\supseteq$. For each $U \in \Delta$, take $e_U \in C_{00}^+(G)$ with support within U and $\|e_U\| = 1$. Then $\{e_U\}_{U\in\Delta}$ is a bounded approximate identity for $L^1(G)$, a closed ideal of $M(G)$. Hence, by Lemma 2.1.1, ν has a unique extension to a homomorphism $\tilde{\nu} : M(G) \to \mathfrak{A}$, with $e = \tilde{\nu}(\delta_e) = \lim_{U\in\Delta} \nu(e_U)$, $\|\tilde{\nu}\| = \|\nu\|$, and $\overline{\nu(L^1(G))} = \overline{\tilde{\nu}(M(G))}$.

It is clear that $\overline{\tilde{\nu}(\ell^1(G))} \subseteq \overline{\tilde{\nu}(M(G))}$, so it remains to prove the inclusion $\nu(L^1(G)) \subseteq \overline{\tilde{\nu}(\ell^1(G))}$. For this it suffices to prove that $\nu(C_{00}^+(G)) \subseteq \overline{\tilde{\nu}(\ell^1(G))}$.

This can be achieved using a portion of the proof of existence and uniqueness of Haar measure, as given in [20, 15.5-6]. (cf. [41, Lemma 2.1].) It follows from [20, 15.6 III], and the subsequent definition of the Haar integral, that for $f \in C_{00}^+(G)$ and $\varepsilon > 0$, there exists $U \in \Delta$ such that if $g \in C_{00}^+(G)$ is zero off U with $\|g\| = 1$, then there exists $h \in \ell^1(G)$ with $\|h\| \leq \|f\|$ and $\|f - h * g\| < \varepsilon$. Take $V \in \Delta$ with $V \subseteq U$, and $\|\nu(e_V) - e\| < \varepsilon$. Then $e_V \in C_{00}^+(G)$ is zero off U, so we can take $h \in \ell^1(G)$ with $\|f - h * e_V\| < \varepsilon$. Then

$$\begin{aligned}\|\nu(f) - \tilde{\nu}(h)\| &\leq \|\nu(f - h * e_V)\| + \|\tilde{\nu}(h)(\nu(e_V) - e)\| \\ &\leq \|\nu\|\|f - h * e_V\| + \|\tilde{\nu}\|\|h\|\|\nu(e_V) - e\| \\ &< \|\nu\|\varepsilon + \|\nu\|\|f\|\varepsilon.\end{aligned}$$

Hence $\nu(f) \in \overline{\tilde{\nu}(\ell^1(G))}$. ∎

2.1.3. Corollary. *A commutative unital Banach algebra $\mathfrak{A}$ has property* (G) *if and only if there is a discrete Abelian group G and a continuous dense-ranged homomorphism $\ell^1(G) \to \mathfrak{A}$.*

Proof. Suppose $\mathfrak{A}$ is a commutative unital Banach algebra $\mathfrak{A}$ with property (G), then by Proposition 1.1.2, there is a locally compact Abelian group G and a continuous dense-ranged homomorphism $\nu : L^1(G) \to \mathfrak{A}$. If we then apply Proposition 2.1.2, the homomorphism $\tilde{\nu} : M(G) \to \mathfrak{A}$ has $\tilde{\nu}|_{\ell^1(G)} : \ell^1(G) \to \mathfrak{A}$ a dense-ranged homomorphism. The converse is clear. ∎

It looks as though we can use Proposition 2.1.2 in a similar way to show that when assessing a unital Banach algebra $\mathfrak{A}$ for property (G), we can restrict our attention to discrete groups. This however, is not the case—it could occur that there is a dense-ranged homomorphism $L^1(G) \to \mathfrak{A}$, where G is amenable, but G_d is not, so that the resulting dense-ranged homomorphism $\ell^1(G) \to \mathfrak{A}$ is not from an amenable Banach algebra.

2.2. A Necessary Condition for Property (G) in Unital Banach Algebras

In the following, $\mathcal{Z}(\mathfrak{A})$ is the *centre* of $\mathfrak{A}$, that is,

$$\mathcal{Z}(\mathfrak{A}) = \{a \in \mathfrak{A} : ab = ba, (b \in \mathfrak{A})\}.$$

2.2.1. Theorem. *Suppose $\mathfrak{A}$ is a unital Banach algebra with property* (G). *Then*

$$\overline{\operatorname{span}}\{ab - ba : a, b \in \mathfrak{A}\} \cap \mathcal{Z}(\mathfrak{A}) = \{0\}.$$

Proof. Let G be an amenable locally compact group and $\nu : L^1(G) \to \mathfrak{A}$ be a dense-ranged homomorphism. Since $\mathfrak{A}^{-1}$ is open, $\operatorname{rng} \nu \cap \mathfrak{A}^{-1} \neq \varnothing$, and we can apply Lemma 2.1.2 to obtain an extension $\tilde{\nu} : M(G) \to \mathfrak{A}$ with $\|\tilde{\nu}\| = \|\nu\|$, $\mathfrak{A} = \overline{\tilde{\nu}(\ell^1(G))}$

and $\tilde{\nu}(\delta_e) = e$. Then

$$\begin{aligned}\overline{\operatorname{span}}\{ab - ba : a, b \in \mathfrak{A}\} &= \overline{\operatorname{span}}\{a\tilde{\nu}(f) - \tilde{\nu}(f)a : a \in \mathfrak{A}, f \in \ell^1(G)\}\\ &= \overline{\operatorname{span}}\{a\tilde{\nu}(\delta_x) - \tilde{\nu}(\delta_x)a : a \in \mathfrak{A}, x \in G\}\\ &= \overline{\operatorname{span}}\{\tilde{\nu}(\delta_{x^{-1}})a\tilde{\nu}(\delta_x) - a : a \in \mathfrak{A}, x \in G\}.\end{aligned}$$

Thus it suffices to show that for each $z \in \mathcal{Z}(\mathfrak{A})$, there is an element of $\mathfrak{A}^*$ that annihilates each $\tilde{\nu}(\delta_{x^{-1}})a\tilde{\nu}(\delta_x) - a$, but not z.

Take $z \in \mathcal{Z}(\mathfrak{A})$. Let $\psi \in \mathfrak{A}^*$ be such that $\psi(z) \neq 0$ and $\|\psi\| < 1$. For each $a \in \mathfrak{A}$, define the function ψ_a on G by $\psi_a(x) = \psi\big(\tilde{\nu}(\delta_{x^{-1}})a\tilde{\nu}(\delta_x)\big)$, $(x \in G)$. Then $\sup_{x\in G} |\psi_a(x)| \leq \|\nu\|^2\|a\|$, so $\psi_a \in \ell^\infty(G)$. Define $\Psi : \mathfrak{A} \to \ell^\infty(G)$ by $\Psi(a) = \psi_a$. Then Ψ is linear with $\|\Psi\| \leq \|\nu\|^2$. If $a \in \operatorname{rng} \nu$, say $a = \nu(f)$, then for each $x \in G$, $\psi_a(x) = \psi \circ \nu(\delta_{x^{-1}} * f * \delta_x)$, so $\psi_a \in C_b(G)$. Hence $\Psi(\operatorname{rng} \nu) \subseteq C_b(G)$, a closed subalgebra of $\ell^\infty(G)$, and since Ψ is continuous, $\Psi(\mathfrak{A}) \subseteq C_b(G)$.

Now, if $a \in \mathfrak{A}$ and $x, y \in G$, then

$${}_y\big(\Psi(a)\big)(x) = \psi_a(yx) = \psi\big(\,\tilde{\nu}(\delta_{x^{-1}})\tilde{\nu}(\delta_{y^{-1}})a\tilde{\nu}(\delta_y)\tilde{\nu}(\delta_x)\,\big) = \Psi\big(\,\tilde{\nu}(\delta_{y^{-1}})a\tilde{\nu}(\delta_y)\,\big)(x).$$

So that if M is a left-invariant mean on $C_b(G)$, then

$$M \circ \Psi(a) = M\big({}_y\Psi(a)\big) = M \circ \Psi\big(\,\tilde{\nu}(\delta_{y^{-1}})a\tilde{\nu}(\delta_y)\big).$$

Hence $M \circ \Psi \in \mathfrak{A}^*$ annihilates each $\tilde{\nu}(\delta_{y^{-1}})a\tilde{\nu}(\delta_y) - a$. But

$$\psi_z(x) = \psi\big(\tilde{\nu}(\delta_{x^{-1}})z\tilde{\nu}(\delta_x)\big) = \psi\big(\tilde{\nu}(\delta_{x^{-1}})\tilde{\nu}(\delta_x)z\big) = \psi(z),$$

so that $\Psi(z)$ is the constant function $\psi(z)$. Hence $M \circ \Psi(z) = \psi(z) \neq 0$. ∎

2.3. The Cuntz Algebras

Suppose H is a separable infinite-dimensional Hilbert space, n is an integer greater than or equal to 2, and $H_1, \ldots, H_n$ are orthogonal closed infinite-dimensional subspaces of H such that $H_1 + \cdots + H_n = H$. For each $1 \leq k \leq n$, let S_k be a linear isometry from H onto H_k. Then $S_1, \ldots, S_n \in \mathcal{B}(H)$ and $I = S_1^*S_1 = \cdots = S_n^*S_n = S_1S_1^* + \cdots + S_nS_n^*$. Let $\mathcal{O}_n$ be the C*-algebra generated by $S_1, \ldots, S_n$, which we call the *Cuntz algebra on n generators*. This algebra was introduced by J. Cuntz in [12], where it is shown not to depend on the actual isometries $S_1, \ldots, S_n$ chosen, but only on n. In [35], it is shown that the Cuntz algebras are amenable. However,

$$(S_1^*S_1 - S_1S_1^*) + \cdots + (S_n^*S_n - S_nS_n^*) = (n-1)I \in \mathcal{Z}(\mathcal{O}_n),$$

so we see that $\mathcal{O}_n$ cannot have property **(G)**.

This seems related to other properties of the Cuntz algebras related to amenability. In particular, the Cuntz algebras are amenable, but not strongly amenable. (Strong amenability is a property of C^*-algebras defined in [23]. The Cuntz algebras were shown to not be strongly amenable in [35].) This absence of strong amenability in the Cuntz algebras can be partially related to the absence of property (G), as follows.

Suppose $\mathfrak{A}$ is a C^*-subalgebra of $\mathcal{B}(H)$ with property (G), so that there is an amenable group G and a continuous homomorphism $\nu : L^1(G) \to \mathfrak{A}$ with $\operatorname{rng} \nu \cap \mathfrak{A}^{-1} \neq \varnothing$. By Proposition 2.1.2, we have a homomorphism $\ell^1(G) \to \mathfrak{A}$, which gives a continuous representation $\pi : G \to \mathcal{B}(H)$ with $\pi(x) \leq \|\nu\|$, for each $x \in G$. (cf. [33, p.77].) Then by [33, Corollary 17.6], π is equivalent to a unitary representation, that is, there is an isomorphism $\Psi : H \to H$ such that $\pi' : x \mapsto \Psi^{-1}\pi(x)\Psi$ is a continuous representation of G with each $\pi'(x)$ unitary. By [23, Proposition 7.8], $\mathfrak{A}' = \overline{\operatorname{span}}\, \pi'(G)$ is a strongly amenable Banach algebra. Moreover, $\Psi\mathfrak{A}'\Psi^{-1} = \overline{\operatorname{span}}\big(\Psi\pi'(G)\Psi^{-1}\big) = \overline{\operatorname{span}}\, \pi(G) = \mathfrak{A}$. Now, if strong amenability was preserved by this transformation $\mathfrak{A}' \mapsto \Psi\mathfrak{A}'\Psi^{-1}$, then we could conclude that $\mathfrak{A}$ is strongly amenable. This would provide a more direct proof that the Cuntz algebras do not have property **(G)**. Unfortunately, it is not clear whether strong amenability in C^*-algebras is preserved by this transformation, so this avenue is not open to us.

let S_k be a linear isometry from H onto H_k. Then $S_1, \ldots, S_n \in \mathcal{B}(H)$ and $I = S_1S_1^* + \cdots + S_nS_n^*$. Let $\mathcal{O}_n$ be the C^*-algebra generated by $S_1, \ldots, S_n$, which we call the *Cuntz algebra on n generators*. This algebra was introduced by J. Cuntz in [12], where it is shown not to depend on the actual isometries $S_1, \ldots, S_n$ chosen, but only on n. In [35], it is shown that the Cuntz algebras are amenable. However,

$$(S_1^*S_1 - S_1S_1^*) + \cdots + (S_n^*S_n - S_nS_n^*) = (n-1)I \in \mathfrak{C}(\mathcal{O}_n)$$

so we see that $\mathcal{O}_n$ cannot have property (G).

This seems related to other properties of the Cuntz algebras related to amenability. In particular, the Cuntz algebras are amenable, but not strongly amenable. (Strong amenability is a property of C^*-algebras defined in [23]. The Cuntz algebras were shown to not be strongly amenable in [35].) This absence of strong amenability in the Cuntz algebras can be partially related to the absence of property (G), as follows.

Suppose $\mathfrak{A}$ is a C^*-subalgebra of $\mathcal{B}(H)$ with property (G), so that there is an amenable group G and a continuous homomorphism $\nu : L^1(G) \to \mathfrak{A}$ with $\operatorname{rng}\nu \cap \mathfrak{A}^{-1} \neq \varnothing$. By Proposition 2.1.2, we have a homomorphism $\ell^1(G) \to \mathfrak{A}$, which gives a continuous representation $\pi : G \to \mathcal{B}(H)$ with $\pi(x) \leq \|\nu\|$, for each $x \in G$. (cf. [33, § 17].) Then by [33, Corollary 17.6], π is equivalent to a unitary representation, that is, there is an isomorphism $\Psi : H \to H$ such that $\pi' : x \mapsto \Psi^{-1}\pi(x)\Psi$ is a continuous representation of G with each $\pi'(x)$ unitary. By [23, Proposition 7.8], $\mathfrak{A}' = \overline{\operatorname{span}}\,\pi'(G)$ is a strongly amenable Banach algebra. Moreover, $\Psi\mathfrak{A}'\Psi^{-1} = \overline{\operatorname{span}}(\Psi\pi'(G)\Psi^{-1}) = \overline{\operatorname{span}}\,\pi(G) = \mathfrak{A}$. Now, if strong amenability was preserved by this transformation $\mathfrak{A}' \mapsto \Psi\mathfrak{A}'\Psi^{-1}$, then we could conclude that $\mathfrak{A}$ is strongly amenable. This would provide a more direct proof that the Cuntz algebras do not have property (G). Unfortunately, it is not clear whether strong amenability of C^*-algebras is preserved by this transformation, so this avenue is not open to us.

Chapter 3. Other Constructions Preserving Amenability

We have seen that the use of dense ranged homomorphisms is not sufficient to bridge the gap between amenability in group algebras and amenability in other Banach algebras. This chapter deals with the possibility that other constructions which preserve amenability could be used in a characterization of Banach algebra amenability. We start with one of the more basic constructions, which is immediately accessible, using methods developed in Chapters 1 and 2. We then undertake an examination of certain aspects of Banach algebra amenability, in order to execute a more intricate construction.

3.1. Amenable Quotients by Amenable Ideals

We have, by [23, Proposition 5.1], that if a Banach algebra $\mathfrak{A}$ has a closed ideal $\mathcal{I}$ such that $\mathfrak{A}/\mathcal{I}$ and $\mathcal{I}$ are both amenable, then $\mathfrak{A}$ is amenable. Given the charter of this chapter, it is natural to consider the following property of Banach algebras.

3.1.1. Definition. We say a Banach algebra $\mathfrak{A}$ has *property* (G$'$) if there are closed subalgebras $\{0\} = \mathfrak{A}_0 \subset \mathfrak{A}_1 \subset \cdots \subset \mathfrak{A}_n = \mathfrak{A}$ such that for each $1 \leq k \leq n$, $\mathfrak{A}_{k-1}$ is a closed ideal of $\mathfrak{A}_k$ and $\mathfrak{A}_k/\mathfrak{A}_{k-1}$ has property (G).

Remark. Each algebra $\mathfrak{A}_k$ has a bounded approximate identity and hence factors. (That is, each $a \in \mathfrak{A}_k$ is a product $a = bc$, for some $b, c \in \mathfrak{A}_k$.) Thus, for any $a \in \mathfrak{A}_k$, there are $a_k, \ldots, a_{n-1} \in \mathfrak{A}_k$ with $a = a_k \ldots a_{n-1}$. Then each $a_j \in \mathfrak{A}_j$, and so if $b \in \mathfrak{A}$, then $a_{n-1}b \in \mathfrak{A}_{n-1}$, $a_{n-2}a_{n-1}b \in \mathfrak{A}_{n-2}$, and so on, eventually giving $ab \in \mathfrak{A}_k$. Hence $\mathfrak{A}_k$ is a right ideal of $\mathfrak{A}$. We can similarly show that $\mathfrak{A}_k$ is a left ideal of $\mathfrak{A}$. Hence each $\mathfrak{A}_k$ is a closed ideal of $\mathfrak{A}$.

This following example shows that property (G$'$) is strictly more general that property (G).

3.1.2. Example. Let G be a nondiscrete locally compact Abelian group, and put $\mathfrak{A} = L^1(G)^\sharp$, the unitization of $L^1(G)$. Then $L^1(G)$ is an ideal of $\mathfrak{A}$ with $\mathfrak{A}/\mathcal{I} \cong \mathbb{C} \cong \ell^1(\{e\})$, so that $\mathfrak{A}$ has property **(G′)**. Note that $\mathfrak{A}$ can be regarded as the closed subalgebra $L^1(G) + \mathbb{C}\delta_e$ of $M(G)$. Supposing $\mathfrak{A}$ has property **(G)**, then by Corollary 2.1.3, there is a discrete Abelian group G' and a homomorphism $\nu : \ell^1(G') \to M(G)$ with $\overline{\operatorname{rng} \nu} = L^1(G) + \mathbb{C}\delta_e$. Clearly, $\Phi_{\mathfrak{A}}$ can be identified with $\Gamma \cup \{\varphi_\infty\}$, the one-point compactification of $\Phi_{L^1(G)} \cong \Gamma$. If we now apply each of Theorems 1.2.1 and 1.3.3 to the homomorphism ν, we obtain a proper continuous injection $\alpha : \Phi_{\mathfrak{A}} \to \Gamma'$ such that $\alpha|_\Gamma$ is a non-proper piecewise affine map. Thus $\alpha(\Phi_{\mathfrak{A}})$ is closed, and by continuity, $\alpha(\varphi_\infty) \in \overline{\alpha(\Gamma)}$. By Lemma 1.4.1, there exists $S \in \mathcal{R}_0(\Gamma)$ such that $\alpha(\varphi_\infty) \in \overline{\alpha(S)}$ and $\alpha|_S$ has an affine extension $\alpha' : E_0(S) \to \Gamma'$. Put $E = E_0(S)$, then $\alpha(S)$ is not closed in Γ', but it is clopen in $\alpha'(E)$, so that $\alpha'(E)$, a coset, is not closed in Γ'. Thus $\overline{\alpha'(E)} \setminus \alpha'(E)$ is dense in $\overline{\alpha'(E)}$, and since $\overline{\alpha(S)}$ is clopen in $\overline{\alpha'(E)}$, $\overline{\alpha(S)} \setminus \alpha(S)$ is dense in $\overline{\alpha(S)}$. However, $\overline{\alpha(S)} \setminus \alpha(S) = \{\varphi_\infty\}$, which is absurd.

Examples with more complicated quotient algebras are easily constructed. For instance, take any $\varphi \in \Phi_{L^1(\mathbb{R})}$, and put $\mathfrak{A} = L^1(\mathbb{R}) \times L^1(\mathbb{R})$ with $\|(f,g)\| = \|f\| + \|g\|$ and $(f_1, g_1)(f_2, g_2) = (f_1 * f_2 + \varphi(g_1)f_2 + \varphi(g_2)f_1, g_1 * g_2)$. Then $\mathcal{I} = L^1(\mathbb{R}) \times \{0\}$ is a closed ideal of $\mathfrak{A}$ with $\mathcal{I} \cong \mathfrak{A}/\mathcal{I} \cong L^1(\mathbb{R})$. To show that $\mathfrak{A}$ lacks property **(G)**, we have an epimorphism $I \times \varphi : L^1(\mathbb{R}) \times L^1(\mathbb{R}) \to L^1(\mathbb{R}) \times \mathbb{C} = L^1(\mathbb{R})^\sharp$, and we have seen above that $L^1(\mathbb{R})^\sharp$ lacks property **(G)**.

3.1.3. Proposition. *Suppose $n > 0$ and H is a finite group of automorphisms of $\mathbb{R}^n$. Then $L^1_H(\mathbb{R}^n)$ is amenable but does not have property* **(G′)**.

Proof. By Proposition 1.7.3, $L^1_H(\mathbb{R}^n)$ is amenable, whereas by Proposition 1.1.2 and Corollary 1.3.6, there is no nonzero homomorphism from a group algebra into $L^1_H(\mathbb{R}^n)$, so we cannot even get $\mathfrak{A}_1$ as in the definition of property **(G′)**. ∎

3.1.4. Proposition. *If $n \geq 2$, then the Cuntz algebra $\mathcal{O}_n$ does not have property* **(G′)**.

Proof. By [12, Theorem 1.13], $\mathcal{O}_n$ is simple, and so the only chain of ideals $\{0\} = \mathfrak{A}_0 \subset \mathfrak{A}_1 \subset \cdots \subset \mathfrak{A}_m = \mathcal{O}_n$ is the trivial one with $m = 1$. Then since $\mathfrak{A}_1/\mathfrak{A}_0 = \mathcal{O}_n$ lacks property (G), $\mathcal{O}_n$ cannot possibly have property (G′). ∎

3.2. Quantifying Amenability

We know that the direct sum of a pair of amenable Banach algebras is amenable, and that the direct sum of a pair of Banach algebras with property (G) has property (G). The case for infinite direct sums is not so straightforward—we will encounter infinite direct sums of amenable Banach algebras that are not amenable. The purpose of this section is to examine certain aspects of amenability so that we can determine whether infinite direct sums, and similar constructions, retain the amenability present in the component parts.

3.2.1. Definition. A Banach algebra $\mathfrak{A}$ is *M-amenable* for $M > 0$ if there is an approximate diagonal for $\mathfrak{A}$ bounded by M.

3.2.2. Lemma. *Suppose $\mathfrak{A}$ is a Banach algebra and $\{d_n\}_{n\in\Delta}$ is a bounded net in $\mathfrak{A} \hat{\otimes} \mathfrak{A}$. Then $\mathfrak{A}_0 = \{a \in \mathfrak{A} : \pi(d_n)a \to a \text{ and } d_n \cdot a - a \cdot d_n \to 0\}$ is a closed subalgebra of $\mathfrak{A}$.*

Proof. Suppose $a, b \in \mathfrak{A}_0$, then $\pi(d_n)ab - ab = (\pi(d_n)a - a)b \to 0$ and $d_n \cdot ab - ab \cdot d_n = (d_n \cdot a - a \cdot d_n) \cdot b + a \cdot (d_n \cdot b - b \cdot d_n) \to 0$, so that $\mathfrak{A}_0$ is a subalgebra of $\mathfrak{A}$. Let M be a bound for $\{d_n\}_{n\in\Delta}$. For $a \in \overline{\mathfrak{A}_0}$ and $\varepsilon > 0$, take $a_0 \in \mathfrak{A}_0$ with $\|a - a_0\| < \varepsilon$. Since $a_0 \in \mathfrak{A}$, there exists $n_0 \in \Delta$ with $n \geq n_0 \implies \|\pi(d_n)a_0 - a_0\| < \varepsilon$ and $\|d_n \cdot a_0 - a_0 \cdot d_n\| < \varepsilon$. Then for $n > n_0$,

$$\begin{aligned}\|\pi(d_n)a - a\| &\leq \|\pi(d_n)\|\|a - a_0\| + \|\pi(d_n)a_0 - a_0\| + \|a_0 - a\| \\ &< (M + 2)\varepsilon\end{aligned}$$

$$\begin{aligned}\text{and } \|d_n \cdot a - a \cdot d_n\| &\leq \|d_n\|\|a - a_0\| + \|d_n \cdot a_0 - a_0 \cdot d_n\| + \|a - a_0\|\|d_n\| \\ &< (2M + 1)\varepsilon.\end{aligned}$$

Hence $a \in \mathfrak{A}_0$, so that $\mathfrak{A}_0$ is closed. ∎

3.2.3. Lemma. *If $\{d_n\}_{n\in\Delta}$ is an approximate diagonal for a Banach algebra $\mathfrak{A}$ and $\nu : \mathfrak{A} \to \mathfrak{B}$ is a dense-ranged homomorphism, then $\{(\nu \otimes \nu)(d_n)\}_{n\in\Delta}$ is an approximate diagonal for $\mathfrak{B}$.*

Proof. For each $n \in \Delta$, put $d'_n = (\nu \otimes \nu)(d_n)$, then $\{d'_n\}_{n\in\Delta}$ is a bounded net in $\mathfrak{B} \otimes \mathfrak{B}$. By Lemma 3.2.2, it is sufficient to show that $\pi(d'_n)b \to b$ and $d'_n \cdot b - b \cdot d'_n \to 0$ for $b = \nu(a) \in \operatorname{rng} \nu$. Clearly, for each $u, v, w \in \mathfrak{A}$, we have $\pi((\nu \otimes \nu)(u \otimes v)) = \nu(\pi(u \otimes v))$, $(\nu \otimes \nu)(u \otimes v) \cdot \nu(w) = (\nu \otimes \nu)(u \otimes vw)$, and $\nu(w) \cdot (\nu \otimes \nu)(u \otimes v) = (\nu \otimes \nu)(wu \otimes v)$. Hence $\pi(d'_n)b - b = \nu(\pi(d_n)a - a) \to 0$ and $d'_n \cdot b - b \cdot d'_n = \nu \otimes \nu(d_n \cdot a - a \cdot d_n) \to 0$. ■

3.2.4. Lemma. *Suppose $\mathfrak{A}$ and $\mathfrak{B}$ are Banach algebras and $\{c_n\}_{n\in\Delta_1} \subseteq \mathfrak{A} \hat{\otimes} \mathfrak{A}$, $\{d_m\}_{m\in\Delta_2} \subseteq \mathfrak{B} \hat{\otimes} \mathfrak{B}$ are nets. Then $\{c_n + d_m\}_{(n,m)\in\Delta_1\times\Delta_2}$ is an approximate diagonal for $\mathfrak{A} \oplus \mathfrak{B}$ if and only if $\{c_n\}_{n\in\Delta_1}$ and $\{d_m\}_{m\in\Delta_2}$ are approximate diagonals for $\mathfrak{A}$ and $\mathfrak{B}$, respectively.*

Proof. Suppose $\{c_n + d_m\}_{(n,m)\in\Delta_1\times\Delta_2}$ is an approximate diagonal for $\mathfrak{A} \oplus \mathfrak{B}$. Let $P : \mathfrak{A} \oplus \mathfrak{B} \to \mathfrak{A}$ be the natural projection. Then for each n, m, $P \otimes P(c_n + d_m) = c_n$, so by Lemma 3.2.3, $\{c_n\}_{(n,m)\in\Delta_1\times\Delta_2}$ is an approximate diagonal for $\mathfrak{A}$, from which it follows that $\{c_n\}_{n\in\Delta_1}$ is an approximate diagonal for $\mathfrak{A}$. Similarly, $\{d_m\}_{m\in\Delta_2}$ is an approximate diagonal for $\mathfrak{B}$.

Conversely, suppose $\{c_n\}_{n\in\Delta_1}$ and $\{d_m\}_{m\in\Delta_2}$ are approximate diagonals for $\mathfrak{A}$ and $\mathfrak{B}$ respectively. Clearly $\{c_n + d_m\}_{\Delta_1\times\Delta_2}$ is a bounded net in $(\mathfrak{A} \oplus \mathfrak{B}) \hat{\otimes} (\mathfrak{A} \oplus \mathfrak{B})$. Suppose $a \in \mathfrak{A}$, then since each $\pi(d_m) \in \mathfrak{B}$, $\pi(d_m)a = 0$, and $\pi(c_n + d_m)a = \pi(c_n)a \to a$. Also, for each $m \in \Delta_2$, $d_m \cdot a = a \cdot d_m = 0$, so

$$(c_n + d_m) \cdot a - a \cdot (c_n + d_m) = c_n \cdot a - a \cdot c_n \to 0.$$

The corresponding observations for $b \in \mathfrak{B}$ complete the proof. ■

We now present some examples and results concerning infinite direct sums of amenable group algebras. As these are based on finite-dimensional complex algebras, it is pertinent to recall Proposition 0.2.3, which stated that a finite dimensional complex algebra $\mathfrak{A}$ is amenable if and only if $\mathfrak{A}$ is isomorphic to a finite direct sum of matrix algebras. Hence, if such $\mathfrak{A}$ is commutative, then $\mathfrak{A}$ is isomorphic to $\mathbb{C}^n$, where $n = \dim \mathfrak{A}$. Moreover, $\mathbb{C}$ has unique diagonal $e \otimes e$, and so by Lemma 3.2.4, $\mathfrak{A} \cong \mathbb{C}^n$ has unique diagonal. The converse to this also applies—in [27], it was shown that if $n \geq 2$, then there are many diagonals (or splitting idempotents) for $M_n(\mathbb{C})$, and so that if a finite dimensional algebra $\mathfrak{A}$ has unique diagonal, then $\mathfrak{A}$ is commutative.

3.2.5. Example. For $k > 0$, put $\mathfrak{A}_k = \ell^1(\mathbb{Z}_2, \omega_k)$, where $\omega_k(0) = 1$ and $\omega_k(1) = k$. Clearly $\mathfrak{A}_k$ is amenable, with unique diagonal $d_k = \frac{1}{2}(\delta_0 \otimes \delta_0 + \delta_1 \otimes \delta_1)$. Also, since $\ell^1(\mathbb{Z}_2, \omega_k) \hat{\otimes} \ell^1(\mathbb{Z}_2, \omega_k) \cong \ell^1(\mathbb{Z}_2^2, \omega_k \otimes \omega_k)$, $\|d_k\|_{\mathfrak{A}_k} = \frac{1}{2}(k^2 + 1)$. Let $\mathfrak{A} = \bigoplus_{1}^{\infty}{}_0 \mathfrak{A}_k$. Suppose $\mathfrak{A}$ is M-amenable, and take $k > M$. The projection $P_k : \mathfrak{A} \to \mathfrak{A}_k$ has $\|P_k\| = 1$, so by Lemma 3.2.3, $\mathfrak{A}_k$ has an approximate diagonal, bounded by M. Such an approximate diagonal will have a cluster point, which will be a diagonal of norm at most M. However, $\mathfrak{A}_k$ has a unique diagonal, which has norm $\frac{1}{2}(k^2 + 1) > M$. Thus $\mathfrak{A}$ is not M-amenable for any M.

3.2.6. Example. Put $\mathfrak{A} = \bigoplus_{1}^{\infty}{}_1 \ell^1(\mathbb{Z}_2)$, then for each $n > 0$, $\mathfrak{A}_n = \bigoplus_{1}^{n}{}_1 \ell^1(\mathbb{Z}_2)$ has a unique diagonal given by taking $d = \frac{1}{2}(\delta_0 \otimes \delta_0 + \delta_1 \otimes \delta_1)$ and applying Lemma 3.2.4. However, $\mathfrak{A}_n$ is linearly isometric to $\ell^1(\{1, \ldots, n\}) \hat{\otimes} \ell^1(\mathbb{Z}_2) \cong \ell^1(\{1, \ldots, n\} \times \mathbb{Z}_2)$, and so we have that $\mathfrak{A}_n \hat{\otimes} \mathfrak{A}_n$ is linearly isometric to $\ell^1(\{1, \ldots, n\}^2 \times \mathbb{Z}_2^2)$. From this, it is straightforward to show that the unique diagonal for $\mathfrak{A}_n$ has norm n, and so as above, $\mathfrak{A}$ is not amenable. Similarly, we can show that $\bigoplus_{1}^{\infty}{}_p \ell^1(\mathbb{Z}_2)$ is not amenable.

So it appears that we need to take care when considering the amenability of infinite direct sums—the only viable situation appears to be the c_0-direct sum of a family of Banach algebras, each with an approximate diagonal bounded by a given, fixed value. We will see that this construction does, in fact, yield an amenable Banach algebra.

We start by examining the projective norm in the tensor product of a c_0-direct sum of two Banach spaces. In the proof, we use a "partitioning" procedure to compare two elements in a tensor product space. Consider two vectors $x = \kappa_1 x_1 + \cdots + \kappa_m x_m$ and $y = \lambda_1 y_1 + \cdots + \lambda_n y_n$, in some complex vector space, where each $\kappa_j \geq 0$, each $\lambda_k \geq 0$, and $\kappa = \sum_1^m \kappa_j \geq \sum_1^n \lambda_k = \lambda$. We want to be able to write $x = \sum_1^N \kappa_i' x_i'$ and $y = \sum_1^N \lambda_i' y_i'$, where each $\kappa_i' \geq 0$, each x_i' is one of $x_1, \ldots, x_m$, and for each $1 \leq j \leq m$, $\kappa_j = \sum\{\kappa_i' : x_i' = x_j\}$, with similar relations for λ_i' and y_i'. Finally, we require that for each i, $\kappa_i' \geq \lambda_i'$. One approach to this is to set $x_{jk} = x_j$, $y_{jk} = y_k$, $\kappa_{jk} = \kappa_j \lambda_k / \lambda$ and $\lambda_{jk} = \kappa_j \lambda_k / \kappa$, so that $x = \sum_{j,k} \kappa_{jk} x_{jk}$, $y = \sum_{j,k} \lambda_{jk} y_{jk}$, and each $\kappa_{jk} \geq \lambda_{jk}$. Other methods exist—we could take the first pair (κ_1, λ_1), put $\kappa_1' = \lambda_1' = \min\{\kappa_1, \lambda_1\}$, and then repeat this with x and y replaced by $x - \kappa_1' x_1$ and $y - \lambda_1' y_1$, and so on until $y = 0$.

3.2.7. Lemma. *Suppose $\mathfrak{X}_1$ and $\mathfrak{X}_2$ are Banach spaces, and $\mathfrak{X} = \mathfrak{X}_1 \oplus_o \mathfrak{X}_2$. Then the natural injection $\mathfrak{X}_1 \hat{\otimes} \mathfrak{X}_1 \oplus_o \mathfrak{X}_2 \hat{\otimes} \mathfrak{X}_2 \to \mathfrak{X} \hat{\otimes} \mathfrak{X}$ is isometric.*

Proof. It is clearly enough to check that the injection $\mathfrak{X}_1 \otimes \mathfrak{X}_1 \oplus_o \mathfrak{X}_2 \otimes \mathfrak{X}_2 \to \mathfrak{X} \otimes \mathfrak{X}$ is isometric. (Given the projective norm on each tensor product space.)

Let $w_1 \in \mathfrak{X}_1 \otimes \mathfrak{X}_1$ and $w_2 \in \mathfrak{X}_2 \otimes \mathfrak{X}_2$. Suppose $\|w_1\| \geq \|w_2\|$. Let $\varepsilon > 0$, then for each $r = 1, 2$, there exists $\{u_{rk}\}_1^{m_r}, \{v_{rk}\}_1^{m_r} \subseteq \mathfrak{X}_r$ and $\{\lambda_{rk}\}_1^{m_r} \subseteq (0, \infty)$ such that $w_r = \sum_1^{m_r} \lambda_{rk} u_{rk} \otimes v_{rk}$, each $\|u_{rk}\| = \|v_{rk}\| = 1$, and $\sum_1^{m_2} \lambda_{2k} < \sum_1^{m_1} \lambda_{1k} \leq \|w_1\| + \varepsilon$. By partitioning as above, we can assume that $m_1 = m_2$ and that for each $1 \leq k \leq m_1$, $\lambda_{1k} \geq \lambda_{2k}$. Then

$$w_1 + w_2 = \sum_1^{m_1} \Big[(\lambda_{1k} - \lambda_{2k}) u_{1k} \otimes v_{1k} + \frac{\lambda_{2k}}{2} \big((u_{1k} + u_{2k}) \otimes (v_{1k} + v_{2k}) + (u_{1k} - u_{2k}) \otimes (v_{1k} - v_{2k}) \big) \Big],$$

and since each tensor product term in this expression has norm 1, we have

$$\|w_1 + w_2\| \leq \sum_1^{m_1} \Big(|\lambda_{1k} - \lambda_{2k}| + \frac{\lambda_{2k}}{2}(1+1) \Big) = \sum_1^{m_1} \lambda_{1k} \leq \|w_1\| + \varepsilon.$$

Hence $\|w_1 + w_2\| \le \|w_1\|$. Similarly $\|w_1 + w_2\| \le \|w_2\|$ if $\|w_1\| \le \|w_2\|$, and so $\|w_1 + w_2\| \le \max\{\|w_1\|, \|w_2\|\}$.

On the other hand, we have a projection $P_1 : \mathfrak{X} \to \mathfrak{X}_1$ with $\|P_1\| = 1$, and so $P_1 \otimes P_1 : \mathfrak{X} \otimes \mathfrak{X} \to \mathfrak{X}_1 \otimes \mathfrak{X}_1$ has $\|P_1 \otimes P_1\| \le 1$. However, we have $w_1 = P_1 \hat{\otimes} P_1(w_1 + w_2)$, so that $\|w_1\|_{\mathfrak{X}_1 \hat{\otimes} \mathfrak{X}_1} \le \|w_1 + w_2\|_{\mathfrak{X} \hat{\otimes} \mathfrak{X}}$. A similar relation for w_2 gives $\max\{\|w_1\|_{\mathfrak{X}_1 \hat{\otimes} \mathfrak{X}_1}, \|w_2\|_{\mathfrak{X}_2 \hat{\otimes} \mathfrak{X}_2}\} \le \|w_1 + w_2\|_{\mathfrak{X} \hat{\otimes} \mathfrak{X}}$. ∎

Applying this to Lemma 3.2.4 gives the following.

3.2.8. Corollary. *Suppose $M > 0$ and $\mathfrak{A}_1, \ldots, \mathfrak{A}_n$ are M-amenable Banach algebras. Then $\bigoplus_{1}^{n}{}_0\, \mathfrak{A}_k$ is M-amenable.* ∎

If we now want to consider the amenability of a c_0-direct sum of a sequence of amenable Banach algebras, we evidently require that the approximate diagonals have a common bound. Then with the aid of the above corollary, we have an ascending sequence of amenable subalgebras, with a common bound to the approximate diagonals, such that their union is dense. The proposition that follows is then sufficient to show amenability of the direct sum.

3.2.9. Definitions. If $\mathfrak{A}$ is a Banach algebra, a *net of subalgebras* of $\mathfrak{A}$ is a family $\{\mathfrak{A}_n\}_{n\in\Delta}$ of closed subalgebras of $\mathfrak{A}$, indexed by a directed set Δ, such that if $n \le m$, then $\mathfrak{A}_n \subseteq \mathfrak{A}_m$. We say that $\{\mathfrak{A}_n\}_{n\in\Delta}$ is *dense* in $\mathfrak{A}$ if $\bigcup_{n\in\Delta} \mathfrak{A}_n$ is dense in $\mathfrak{A}$.

3.2.10. Proposition. *Suppose $\mathfrak{A}$ is a Banach algebra such that, for some $M > 0$, there is a dense net of M-amenable subalgebras of $\mathfrak{A}$, then $\mathfrak{A}$ is M-amenable.*

Proof. Let $\{\mathfrak{A}_i\}_{i\in\mathbb{I}}$ be a dense net of M-amenable subalgebras of $\mathfrak{A}$, so that for each $i \in \mathbb{I}$, there is a net $\{d_{i,n}\}_{n\in\Delta_i} \subseteq \mathfrak{A}_i \hat{\otimes} \mathfrak{A}_i$, bounded by M, that is an approximate diagonal for $\mathfrak{A}_n$. Now, by the definition of the projective norm, $\{d_{i,n}\}_{m\in\Delta_i}$ as a net in $\mathfrak{A} \hat{\otimes} \mathfrak{A}$, is bounded by M. Put $\Delta = \mathbb{I} \times \prod_{i\in\mathbb{I}} \Delta_i$, with the product order. (That is, $(i, \{n_k\}_{k\in\mathbb{I}}) \le (j, \{m_k\}_{k\in\mathbb{I}})$ if $i \le j$ and each $n_k \le m_k$.) This is clearly a directed set. For $\alpha = (i, \{n_k\}_{k\in\mathbb{I}}) \in \Delta$, put $\tilde{d}_\alpha = d_{i,n_i}$. Then $\{\tilde{d}_\alpha\}_{\alpha\in\Delta}$ is a net in $\mathfrak{A} \hat{\otimes} \mathfrak{A}$, bounded by M.

By Lemma 3.2.2, it suffices to show that $\pi(\tilde{d}_\alpha)a \to a$ and $\tilde{d}_\alpha \cdot a - a \cdot \tilde{d}_\alpha \to 0$ for all $a \in \bigcup_{i\in\mathbb{I}} \mathfrak{A}_i$. Suppose $i \in \mathbb{I}$, $a \in \mathfrak{A}_i$, and $\varepsilon > 0$. For each $k \in \mathbb{I}$ with $k \geq i$, $a \in \mathfrak{A}_k$, so there exists $n_k \in \Delta_k$ such that

$$n > n_k \implies \left\|\pi(d_{k,n})a - a\right\|_{\mathfrak{A}_k} < \varepsilon \text{ and } \left\|d_{k,n} \cdot a - a \cdot d_{k,n}\right\|_{\mathfrak{A}_k\hat{\otimes}\mathfrak{A}_k} < \varepsilon.$$

For all other k (those with $k \not\geq i$), choose $n_k \in \Delta_k$ arbitrarily. Put $\alpha = (i, \{n_k\}_{k\in\mathbb{I}})$. It is now clear that if $\beta = (j, \{m_k\}_{k\in\mathbb{I}}) \geq \alpha$, then

$$\left\|\pi(\tilde{d}_\beta)a - a\right\|_{\mathfrak{A}} = \left\|\pi(d_{j,m_j})a - a\right\|_{\mathfrak{A}_k} < \varepsilon$$

$$\text{and} \qquad \left\|\tilde{d}_\beta \cdot a - a \cdot \tilde{d}_\beta\right\|_{\mathfrak{A}\hat{\otimes}\mathfrak{A}} \leq \left\|d_{j,m_j} \cdot a - a \cdot d_{j,m_j}\right\|_{\mathfrak{A}_k\hat{\otimes}\mathfrak{A}_k} < \varepsilon.$$

Hence $\pi(\tilde{d}_\alpha)a \to a$ and $\tilde{d}_\alpha \cdot a - a \cdot \tilde{d}_\alpha \to 0$, and since each $\left\|\tilde{d}_\alpha\right\| < M$, $\mathfrak{A}$ is M-amenable. ■

3.2.11. Corollary. *If $M > 0$ and $\{\mathfrak{A}_i\}_{i\in\mathbb{I}}$ is a family of M-amenable Banach algebras, then $\mathfrak{A} = \bigoplus_{0,\, i\in\mathbb{I}} \mathfrak{A}_i$ is an M-amenable Banach algebra.*

Proof. Put $\Delta = \{F \subseteq \mathbb{I} : F \text{ is finite}\}$, ordered by $\subseteq$, and for $F \in \Delta$, define $\mathfrak{A}_F = \bigoplus_{0,\, i\in F} \mathfrak{A}_i \subseteq \mathfrak{A}$. By Corollary 3.2.8, each $\mathfrak{A}_F$ is M-amenable. Also, $\bigcup_{F\in\Delta} \mathfrak{A}_F$ is dense in $\mathfrak{A}$, so by Proposition 3.2.10, $\mathfrak{A}$ is M-amenable. ■

Remark. There are other ways in which we could quantify amenability. For example, we could define a Banach algebra $\mathfrak{A}$ to be M-amenable if for any continuous derivation D from $\mathfrak{A}$ into a dual Banach $\mathfrak{A}$-bimodule $\mathfrak{X}^*$, there exists $f \in \mathfrak{X}^*$ such that $D(a) = a \cdot f - f \cdot a$ and $\|f\| \leq M\|D\|$. This is used in [34, Proposition 1.12] to prove a result analogous to Proposition 3.2.10. Consider the proof of [24, Theorem 1.3], where amenability in the cohomological sense is shown to be equivalent to the existence of an approximate diagonal. By keeping track of the norms of the elements involved, it is possible to show that a Banach algebra with an M-bounded approximate diagonal is M-amenable (in the above sense). The converse is not so straightforward—by [23, Proposition 1.6], an M-amenable Banach algebra $\mathfrak{A}$ has 1-sided M-bounded approximate identities (left and right), and consequently $\mathfrak{A}$ has a $(2M + M^2)$-bounded approximate identity. The argument of [24, Theorem 1.3] then gives an approximate diagonal bounded by $(2M + M^2)^2(M+1)$. In the unital case, we have a 1-bounded (approximate) identity, and so we can improve the above bound to $M + 1$. In the commutative case, we have an M-bounded approximate identity, and so the approximate diagonal has bound $M^2(M+1)$. It seems an interesting question to determine whether these bounds can be improved, and by what margin.

Another way in which amenability could be quantified is by specifying a bound on an approximate identity in the diagonal ideal $\ker \pi \subseteq \mathfrak{A} \hat{\otimes} \mathfrak{A}^{\text{op}}$. Unfortunately, it is not quite as easy to relate this bound to a bound on an approximate diagonal, or to the bound on the norm of an element implementing a derivation. The proof of equivalence of amenability to the existence of bounded approximate identity in $\ker \pi$ is quite indirect. (See, for example, [13, Theorem 3.10].)

3.3. Property (G^∞)

We now turn to applying the above concepts to group algebras. We will construct an approximate diagonal for an amenable group algebra in Theorem 3.3.2, but before we do this, we need a better understanding of the relevant algebraic operations on the $L^1(G)$-bimodule $L^1(G \times G) \cong L^1(G) \hat{\otimes} L^1(G)$.

Define T_1, T_2, D to be linear maps $C_0(G \times G) \to C_0(G)$ given by

$$T_1(\psi)(t) = \psi(t, e),$$

$$T_2(\psi)(t) = \psi(e, t),$$

$$\text{and } D(\psi)(t) = \psi(t, t^{-1}),$$

for $\psi \in C_0(G \times G)$. Then $T_1^*, T_2^* : M(G) \to M(G \times G)$ are algebra monomorphisms, each of norm 1 and $D^* : M(G) \to M(G \times G)$ is a linear monomorphism of norm 1.

It can be shown that D is multiplicative if and only if G is Abelian, but we are not concerned with this here. Note also that if $f \in L^1(G)$ and $F \in L^1(G \times G)$, then $T_1^*(f) * F = f \cdot F$ and $F * T_2^*(f) = F \cdot f$.

Recall from [20, Theorems 20.1 & 20.2], that if Δ_G is the modular function on G, then for any $f \in L^1(G)$, $x \in G$,

$$\int_G {}_x f(z) dz = \int_G f(z)\, dz,$$

$$\int_G f_x(z) dz = \Delta_G(x^{-1}) \int_G f(z)\, dz,$$

$$\text{and} \quad \int_G f(z^{-1}) dz = \int_G \Delta_G(z^{-1}) f(z)\, dz.$$

3.3.1. Lemma. *If $f, g \in L^1(G)$, then $T_1^*(f) * D^*(g) \in L^1(G \times G)$ is given almost everywhere by $\big(T_1^*(f) * D^*(g)\big)(s,t) = f(st)g(t^{-1})$ and $D^*(g) * T_2^*(f) \in L^1(G \times G)$ is given almost everywhere by $\big(D^*(f) * T_2^*(g)\big)(s,t) = f(s)g(st)$.*

Proof. By the definition of convolution of measures ([20, Section 19]), we have for each $\psi \in C_0(G \times G)$, that

$$\begin{aligned}
\langle \psi, T_1^*(f) * D^*(g) \rangle &= \int_{G\times G} \int_{G\times G} \psi(ux, vy)\, d\big(T_1^*(f)\big)(u,v)\, d\big(D^*(g)\big)(x,y) \\
&= \int_G \int_G \psi(ux, ex^{-1})\, f(u)\, du\, g(x)\, dx \\
&= \int_G \Delta_G(x^{-1}) \int_G \psi(ux^{-1}, x)\, f(u)\, du\, g(x^{-1})\, dx \\
&= \int_G \int_G \psi(u, x)\, f(ux)\, du\, g(x^{-1})\, dx,
\end{aligned}$$

so that $\big(T_1^*(f) * D^*(g)\big)(s,t) = f(st)g(t^{-1})$ for almost all $s, t \in G$. Also

$$\begin{aligned}
\langle \psi, D^*(f) * T_2^*(g) \rangle &= \int_{G\times G} \int_{G\times G} \psi(ux, vy)\, d\big(D^*(f)\big)(u,v)\, d\big(T_2^*(g)\big)(x,y) \\
&= \int_G \int_G \psi(ue, u^{-1}y) f(u)\, du\, g(y)\, dy \\
&= \int_G \int_G \psi(u, y) f(u) g(uy)\, du\, dy,
\end{aligned}$$

from which the second formula follows. ∎

3.3.2. Theorem. *An amenable group algebra is 1-amenable.*

Proof. Let G be an amenable group. By [30, Theorem 4.16], there exists a summing net for G, that is, a net $\{K_n\}_{n\in\Delta}$ of compact subsets of G such that for any compact $U \subseteq G$, $\lambda(K_n \triangle xK_n)/\lambda(K_n) \to 0$ uniformly for $x \in U$. Also, let $\{e_n\}_{n\in\Delta}$ be a bounded approximate identity for $L^1(G)$ with $\|e_n\| \le 1$ for all $n \in \Delta$. (In general, the index sets will not be the same, but by taking the cartesian product of the index sets, we can assume that they are.)

For each $n \in \Delta$, define $F_n = \frac{1}{\lambda(K_n)}\chi_{K_n} \in L^1(G)$, $\tilde{d}_n = D^*(F_n) * T_2^*(e_n)$, and $d_n = e_n \cdot \tilde{d}_n = T_1^*(e_n) * D^*(F_n) * T_2^*(e_n)$. Then $\|F_n\| = 1$, $\|\tilde{d}_n\| \le 1$ and $\|d_n\| \le 1$. Also

$$\begin{aligned}\pi(\tilde{d}_n)(t) &= \int_G \tilde{d}_n(s, s^{-1}t)\,ds \\ &= \int_G F_n(s)e_n(ss^{-1}t)\,ds \qquad \text{(Lemma 3.3.1)} \\ &= e_n(t),\end{aligned}$$

so that $\pi(\tilde{d}_n) = e_n$ and $\pi(d_n) = e_n * e_n$. Thus $\{\pi(d_n)\}_{n\in\Delta}$ is a bounded approximate identity for $L^1(G)$. Also, if $f \in L^1(G)$, then by Lemma 3.3.1,

$$\begin{aligned}\|d_n \cdot f - f \cdot d_n\| \le{}& \|T_1^*(e_n) * D^*(F_n) * T_2^*(e_n * f - f * e_n)\| \\ &+ \|T_1^*(e_n)\big(D^*(F_n) * T_2^*(f) - T_1^*(f) * D^*(F_n)\big)T_2^*(e_n)\| \\ &+ \|T_1^*(e_n * f - f * e_n) * D^*(F_n) * T_2^*(e_n)\| \\ \le{}& 2\|e_n * f - f * e_n\| + \|D^*(F_n) * T_2^*(f) - T_1^*(f) * D^*(F_n)\|.\end{aligned}$$

Clearly $\|e_n * f - f * e_n\| \to 0$. Now suppose $f = \chi_U$, for U a compact set. Then

$$\begin{aligned}\|D^*(F_n) * T_2^*(f) - T_1^*(f) * D^*(F_n)\| &= \int_G\int_G |F_n(s)f(st) - f(st)F_n(t^{-1})|\,ds\,dt \\ &= \int_G\int_G f(t)|F_n(s) - F_n(t^{-1}s)|\,ds\,dt \\ &= \int_U \frac{\lambda(K_n \triangle tK_n)}{\lambda(K_n)}\,dt \\ &\to 0.\end{aligned}$$

Since the span of such functions f is dense in $L^1(G)$, we have by Lemma 3.2.2 that the net $\{d_n\}_{n\in\Delta}$ is an approximate diagonal for $L^1(G)$. Since each $\|d_n\| \le 1$, $L^1(G)$ is 1-amenable. ∎

3.3.3. Definitions. Suppose $\mathfrak{A}$ is a Banach algebra such that for some $K > 0$ there exists an amenable locally compact group and a dense-ranged homomorphism $\nu : L^1(G) \to \mathfrak{A}$ with $\|\nu\| \leq K$. Then we say that $\mathfrak{A}$ has *property* $(\mathbf{G}_K)$. Supposing $\mathfrak{A}$ has a dense net of subalgebras, each having property $(\mathbf{G}_K)$, for some fixed $K > 0$, then we say that $\mathfrak{A}$ has *property* $(\mathbf{G}^\infty)$.

3.3.4. Proposition. *A Banach algebra $\mathfrak{A}$ with property $(\mathbf{G}_K)$ is K^2-amenable.*

Proof. By hypothesis, there is an amenable group G and a dense-ranged homomorphism $\nu : L^1(G) \to \mathfrak{A}$ with $\|\nu\| \leq K$. By Theorem 3.3.2 there exists a net $\{d_n\}_{n\in\Delta} \subseteq L^1(G) \hat{\otimes} L^1(G)$ that is an approximate diagonal for $L^1(G)$, bounded by 1. By Lemma 3.2.3 $\{\nu \otimes \nu(d_n)\}_{n\in\Delta}$ is an approximate diagonal for $\mathfrak{A}$, and since $\|\nu \otimes \nu\| \leq \|\nu\|^2$, we have that $\{\nu \otimes \nu(d_n)\}_{n\in\Delta}$ is bounded by K^2, so that $\mathfrak{A}$ is K^2-amenable. ∎

3.3.5. Proposition. *A Banach algebra with property $(\mathbf{G}^\infty)$ is amenable.*

Proof. Suppose $\mathfrak{A}$ has property $(\mathbf{G}^\infty)$, so that $\mathfrak{A}$ has a dense net of subalgebras $\{\mathfrak{A}_n\}_{n\in\Delta}$, each $\mathfrak{A}_n$ having property $(\mathbf{G}_K)$. By Proposition 3.3.4, each algebra $\mathfrak{A}_n$ is K^2-amenable, so by Proposition 3.2.10, $\mathfrak{A}$ is also K^2-amenable. ∎

We may now ask if property $(\mathbf{G}^\infty)$ comes closer to characterizing amenability than property $(\mathbf{G})$. In the next section, we consider property $(\mathbf{G}^\infty)$ in closed subalgebras of commutative group algebras.

3.4. Property $(\mathbf{G}^\infty)$ in Group Subalgebras

Again, since we are dealing with homomorphisms from group algebras into commutative Banach algebras, it suffices to consider Abelian groups, and so we again assume additive notation for the groups considered in this section.

We start by considering property $(\mathbf{G}^\infty)$ in a closed ideal of a commutative group algebra. We have seen in Theorem 1.7.2 and the subsequent discussion, that an ideal $\mathcal{I}$ of a commutative group algebra $L^1(G)$ has property **(G)** if and only if there is an idempotent measure $\mu \in M(G)$ such that $f \mapsto f * \mu$ is a homomorphism from $L^1(G)$ onto $\mathcal{I}$. It is clear that such a homomorphism has norm $\|\mu\|$, and so to be able to control the norm of several homomorphisms of this form, we need to control the norm of the idempotent measures. Proposition 3.4.3 below allows us to do exactly that. We start with a useful lemma.

3.4.1. Lemma. *Suppose* $X = \bigcup_1^n X_k$, *where* $X_1, \ldots, X_n \in \mathcal{R}(\Gamma)$. *Then* $\|\chi_X\|_{B(\Gamma)} \leq \prod_1^n(\|\chi_{X_k}\|_{B(\Gamma)} + 1) - 1$.

Proof. If $n = 2$, then $\chi_X = \chi_{X_1} + \chi_{X_2} - \chi_{X_1}\chi_{X_2}$, from which the result follows. The case $n > 2$ follows by induction. ■

3.4.2. Corollary. *If* $S \in \mathcal{R}_0(\Gamma)$, *then* $\|\chi_S\| \leq 2^{N(S)}$.

Proof. Let $n = N(S)$ and suppose $S = E_0 \setminus (\bigcup_1^n E_k)$. Now, if Λ is an open subgroup of Γ, then $H = \operatorname{Ann}\Lambda$ is compact and $\lambda_H \in L^1(H) \subseteq M(G)$. Moreover, $\|\lambda_H\|_{M(G)} = \|\lambda_H\|_{L^1(H)} = 1$ and $\hat{\lambda}_H = \chi_\Lambda$. Then the norm on $B(\Gamma)$ is translation-invariant, so that for any clopen coset E, $\|\chi_E\| = 1$. It follows from Lemma 3.4.1 that $\|\chi_{E_1\cup\cdots\cup E_n}\| \leq 2^n - 1$ and $\|\chi_S\| \leq 2^n$. ■

3.4.3. Proposition. *Suppose $\mathcal{F}$ is a family of subgroups of Γ that is closed under pairwise intersection and $S \in \mathcal{R}_d(\Gamma)$ is such that for each $\Xi \in \mathcal{F}$, $S + \Xi$ is clopen. Then there exists $\Xi' \in \mathcal{F}$ and a bound $N > 0$ such that if $\Xi \in \mathcal{F}$ and $\Xi \subseteq \Xi'$, then* $\|\chi_{S+\Xi}\|_{B(\Gamma)} \leq N$.

Proof. We follow in the steps of the proof of Proposition 1.4.12, with additional emphasis on controlling the index $[\Lambda_0 \cap \Xi : \Lambda_k \cap \Xi]$ occurring therein. Again, by Lemma 1.4.10 we can assume that Γ is discrete.

Consider first $S \in \mathcal{R}_0(\Gamma)$, say $S = E_0 \setminus (\bigcup_1^m E_k)$, and for each $0 \leq k \leq m$ put $\Lambda_k = E_k - E_k$. Fix some $0 \leq k \leq m$. If $\Xi_1, \Xi_2 \in \mathcal{F}$ and $\Xi_1 \subseteq \Xi_2$, then

$$(\Lambda_0 \cap \Xi_1)/(\Lambda_k \cap \Xi_1) = (\Lambda_0 \cap \Xi_1)/(\Lambda_k \cap (\Lambda_0 \cap \Xi_1)) \cong ((\Lambda_0 \cap \Xi_1) + \Lambda_k)/\Lambda_k,$$

and similarly for Ξ_2. (Note that the case $k = 0$ is trivial, but we include it anyway.) Also $((\Lambda_0 \cap \Xi_1) + \Lambda_k)/\Lambda_k \subseteq ((\Lambda_0 \cap \Xi_2) + \Lambda_k)/\Lambda_k$, so that $[\Lambda_0 \cap \Xi_1 : \Lambda_k \cap \Xi_1] \leq [\Lambda_0 \cap \Xi_2 : \Lambda_k \cap \Xi_2]$. Thus $[\Lambda_0 \cap \Xi : \Lambda_k \cap \Xi]$ is a non-decreasing function of $\Xi \in \mathcal{F}$. It follows that either $[\Lambda_0 \cap \Xi : \Lambda_k \cap \Xi]$ is infinite for all $\Xi \in \mathcal{F}$, or there exists $\Xi_k \in \mathcal{F}$ and $n_k \in \mathbb{N}$ such that $\Xi \in \mathcal{F}$ and $\Xi \subseteq \Xi_k$ together imply $[\Lambda_0 \cap \Xi : \Lambda_k \cap \Xi] = n_k$. Let $\mathbb{J} \subseteq \{0, \ldots, n\}$ be the set of all k for which the second of these possibilities occurs. Clearly $0 \in \mathbb{J}$ and $n_0 = 1$.

Put $\Lambda_{\mathbb{J}} = \bigcap_{\mathbb{J}} \Lambda_k$, $S' = E_0 \setminus (\bigcup_{\mathbb{J}\setminus\{0\}} E_k)$ and $\Xi' = \bigcap_{\mathbb{J}} \Xi_k$. (We included $0 \in \mathbb{J}$ to make these definitions easy.) Suppose $\Xi \in \mathcal{F}$ has $\Xi \subseteq \Xi'$, then for each nonzero $k \in \mathbb{J}$, $[\Lambda_0 \cap \Xi : \Lambda_k \cap \Xi] = n_k$ and for all other nonzero k, $[\Lambda_0 \cap \Xi : \Lambda_k \cap \Xi]$ is infinite. Thus, as in Lemma 1.4.12, $S + (\Lambda_0 \cap \Xi) = S' + (\Lambda_0 \cap \Xi)$. Put $n_{\mathbb{J}} = \prod_{\mathbb{J}} n_k$, then $[\Lambda_0 \cap \Xi : \Lambda_{\mathbb{J}} \cap \Xi] \leq n_{\mathbb{J}}$, so we can take $F \subseteq \Lambda_0 \cap \Xi$ with $|F| \leq n_{\mathbb{J}}$ and $(\Lambda_{\mathbb{J}} \cap \Xi) + F = \Lambda_0 \cap \Xi$. Again, $S + (\Lambda_0 \cap \Xi) = S' + F$, so by Lemma 3.4.1 and Corollary 3.4.2,

$$\begin{aligned}\|\chi_{S+(\Lambda_0\cap\Xi)}\| &\leq \prod_{\gamma\in F}(\|\chi_{S'+\gamma}\| + 1) - 1 \\ &\leq (2^{|\mathbb{J}|} + 1)^{n_{\mathbb{J}}} - 1.\end{aligned}$$

Let N be this last number. It is independent of the choice of $\Xi \in \mathcal{F}$ with $\Xi \subseteq \Xi'$. Then, by Lemma 1.4.11, we have

$$\begin{aligned}\|\chi_{S+\Xi}\|_{B(\Gamma)} &= \|\chi_{S+\Xi}\|_{B(E_0+\Xi)} \\ &= \|\chi_{Q_\Xi(S)}\|_{B((E_0+E)/\Xi)} \\ &= \|\chi_{Q_{(\Lambda_0\cap\Xi)}(S)}\|_{B(E_0/(\Lambda_0\cap\Xi))} \\ &= \|\chi_{S+(\Lambda_0\cap\Xi)}\|_{B(E_0)} \\ &= \|\chi_{S+(\Lambda_0\cap\Xi)}\|_{B(\Gamma)} \\ &\leq N.\end{aligned}$$

Now suppose $S \in \mathcal{R}(\Gamma)$, so there exist $S_1, \ldots, S_n \in \mathcal{R}_0(\Gamma)$ with $S = \bigcup_1^n S_k$. Applying the above argument to each S_k, we obtain $\Xi'_k \in \mathcal{F}$ and $N_k > 0$ such that $\Xi \in \mathcal{F}$ and $\Xi \subseteq \Xi'_k$ together imply $\|\chi_{S+\Xi}\| \leq N_k$. Put $\Xi' = \bigcap_1^n \Xi'_k \in \mathcal{F}$ and $N = \prod_1^n (N_k + 1) - 1$, then by Lemma 3.4.1, these are sufficient. ∎

Remark. Although the bound here seems unnecessarily large, it is possible to create examples of $S \in \mathcal{R}_0(\Gamma)$ for which $N(S+\Xi)$ is exponential in $N(S)$, and as we are using $N(S+\Xi)$ to estimate $\|\chi_{S+\Xi}\|$, we need quite a large bound. As an example of this, let $m, n \in \mathbb{N}$, put $\Gamma = \mathbb{Z}^m \times \mathbb{Z}_m$, and $\Xi = 0 \times \mathbb{Z}_m$. Take $\{a_{jk} : 1 \le j \le m, 1 \le k \le n\}$ to be distinct integers, and for $1 \le j \le m$ and $1 \le k \le n$, put $E_{jk} = \mathbb{Z}^{j-1} \times \{a_k\} \times \mathbb{Z}^{m-k} \times \{j\}$. Then with $S = \Gamma \setminus \left(\bigcup_{j,k} E_{j,k}\right)$, $N(S) = nm$. However, $\mathbb{Z}^m \setminus Q_\Xi(S) = \{a_{11}, \ldots, a_{1n}\} \times \cdots \times \{a_{m1}, \ldots, a_{mn}\}$, which contains no cosets larger than singletons, so that $N\big(Q_\Xi(S)\big) = n^m$. It was examples such as this that motivated the method of proof for Propositions 1.4.12 and 3.4.3.

3.4.4. Proposition. *Let G be a locally compact Abelian group and let Ξ be the component of the identity in Γ. An ideal $\mathcal{I} \subseteq L^1(G)$ has property* ($\mathbf{G}^\infty$) *if and only if $S = Z(\mathcal{I}) \in \mathcal{R}_c(\Gamma)$ and $S + \Xi = S$.*

Proof. Suppose $\mathcal{I}$ has property ($\mathbf{G}^\infty$), so that $\mathcal{I}$ is amenable and $S \in \mathcal{R}_c(\Gamma)$, by Theorem 1.7.2. Let $M > 0$ and $\{\mathfrak{A}_n\}_{n\in\Delta}$ be such that each $\mathfrak{A}_n$ is a closed subalgebra of $\mathcal{I}$ with property ($\mathbf{G}_M$) and $\mathcal{I} = \overline{\bigcup_{n\in\Delta} \mathfrak{A}_n}$.

Since each $\mathfrak{A}_n \subseteq L^1(G)$ has property ($\mathbf{G}_M$), we have by Proposition 1.7.1 that there exists a locally compact Abelian group G_n and a proper piecewise affine map α from $Y_n = \Gamma \setminus Z(\mathfrak{A}_n)$ into Γ_n such that $\mathfrak{A}_n = \kappa(\alpha_n)$. Then $S = Z(\mathcal{I}) = \bigcap_{n\in\Delta} Z(\mathfrak{A}_n) = \bigcap_{n\in\Delta} \Gamma\setminus Y_n$. For each $\gamma \in \Gamma$, $\gamma+\Xi$ is connected, and since each Y_n is clopen, either $\gamma+\Xi \subseteq \Gamma\setminus Y_n$ or $\gamma+\Xi \subseteq Y_n$. Hence $S+\Xi = \{\gamma+\Xi : \gamma \in S\} \subseteq \Gamma\setminus Y_n$, so $S+\Xi \subseteq S \subseteq S+\Xi$.

Conversely, suppose $S \in \mathcal{R}_c(\Gamma)$ has $S+\Xi = S$. Let $\mathcal{F}$ be the set of clopen subgroups of Γ. Then by Proposition 3.4.3, there exists $\Lambda_0 \in \mathcal{F}$ and a bound $N > 0$ such that if Λ is a clopen subgroup of Λ_0, then $\|\chi_{S+\Lambda}\|_{B(\Gamma)} < N$. Put $\mathcal{F}' = \{\Lambda \in \mathcal{F} : \Lambda \subseteq \Lambda_0\}$, then for $\Lambda \in \mathcal{F}'$, let $\mu_\Lambda \in M(G)$ be such that $\hat{\mu}_\Lambda = \chi_{\Gamma\setminus(S+\Lambda)}$, and let $\nu_\Lambda : L^1(G) \to L^1(G)$ be the projection $f \mapsto f * \mu_\Lambda$. Then ν_Λ is a homomorphism onto $\mathcal{I}(S+\Lambda)$ with $\|\nu_\Lambda\| = \|\mu_\Lambda\| < N+1$, so that $\mathcal{I}(S+\Lambda)$ has property ($\mathbf{G}_{N+1}$). Clearly $\Xi = \bigcap_{\Lambda\in\mathcal{F}'} \Lambda$, so $\mathcal{I}_0 = \bigcup_{\Lambda\in\mathcal{F}'} \mathcal{I}(S+\Lambda)$ is an ideal of $L^1(G)$ with $Z(\mathcal{I}_0) = \bigcap_{\Lambda\in\mathcal{F}'}(S+\Lambda) = S+\Xi = S$. Since $S \in \mathcal{R}_c(\Gamma)$ is a set of synthesis,

$$\mathcal{I}(S) = \overline{\mathcal{I}_0} = \overline{\bigcup_{\Lambda\in\mathcal{F}'} \mathcal{I}(S+\Lambda)},$$

and since each $\mathcal{I}(S+\Lambda)$ has property $(\mathbf{G}_{N+1})$, $\mathcal{I}(S)$ has property $(\mathbf{G}^\infty)$. ∎

Thus, in the case of group algebra ideals, property $(\mathbf{G}^\infty)$ does occur in cases where property $(\mathbf{G})$ does not. For instance, if $G = \sum_1^\infty \mathbb{Z}_2$, so that $\Gamma = \prod_1^\infty \mathbb{Z}_2$ and $\Xi = \{e\}$, then all amenable ideals in $\ell^1(G)$ (including those of finite codimension) have property $(\mathbf{G}^\infty)$, whereas no finite set in Γ is open, so no ideal of finite codimension has property $(\mathbf{G})$. However, in cases where $\Xi \neq \{e\}$, Ξ will have proper closed subgroups, so that there will be many $S \subseteq \mathcal{R}_c(\Gamma)$ with $S \subset S+\Xi$, and hence many amenable closed ideals of $L^1(G)$ without property $(\mathbf{G}^\infty)$.

It is also possible to show that many of the algebras $L^1_H(G)$ lack property $(\mathbf{G}^\infty)$. For instance, if Γ is connected, then by Corollary 1.3.5, a homomorphism from a group algebra into $L^1(G)$ has range $\{f \in L^1(G) : f = 0 \text{ off } G_0\} \cong L^1(G_0)$, for some clopen subgroup G_0 of G. Now, if $L^1(G_0) \subseteq L^1_{\text{sym}}(G)$, then $x^2 = e$ for each $x \in G_0$. Thus, $\hat{G}_0$ is of bounded order (1 or 2), and since $\Gamma/\operatorname{Ann}(G_0) \cong \hat{G}_0$ is connected, we must have $G_0 = \{e\}$. Thus $L^1_{\text{sym}}(G)$ cannot have property $(\mathbf{G}^\infty)$.

This example also excludes the possibility of building a dense chain of closed algebras $\{0\} = \mathfrak{A}_0 \subset \mathfrak{A}_1 \subset \mathfrak{A}_2 \subset \cdots$ in $\mathfrak{A}$ such that for each $k > 0$, $\mathfrak{A}_{k-1}$ is a closed ideal of $\mathfrak{A}_k$ and $\mathfrak{A}_k/\mathfrak{A}_{k-1}$ has property $(\mathbf{G})$, for we cannot get past the first step of having a nonzero $\mathfrak{A}_1$ with property $(\mathbf{G})$.

3.5. Property $(\mathbf{G}^\infty)$ in Unital Banach Algebras

More interesting, perhaps, is the following extension of Theorem 2.2.1, which excludes the Cuntz algebras $\mathcal{O}_n$ from having property $(\mathbf{G}^\infty)$.

3.5.1. Theorem. *Suppose $\mathfrak{A}$ is a unital Banach algebra with property $(\mathbf{G}^\infty)$, then*

$$\overline{\operatorname{span}}\{ab - ba : a, b \in \mathfrak{A}\} \cap \mathcal{Z}(\mathfrak{A}) = \{0\}.$$

Proof. Suppose $K > 0$ and $\mathfrak{A} = \overline{\bigcup_{n\in\Delta} \mathfrak{A}_n}$, where each $\mathfrak{A}_n$ is a closed subalgebra with property $(\mathbf{G}_K)$ and $n < n' \implies \mathfrak{A}_n \subseteq \mathfrak{A}_{n'}$. Due to this latter property, we will be able to replace $\{\mathfrak{A}_n\}_{n\in\Delta}$ by any subnet and retain the assumed properties.

For each $n \in \Delta$, $\mathfrak{A}_n$ has property $(\mathbf{G}_K)$, so there is an amenable locally compact group G_n and a continuous dense-ranged homomorphism $\nu_n : L^1(G_n) \to \mathfrak{A}_n$ with $\|\nu_n\| \le K$. Take $z \in \mathcal{Z}(\mathfrak{A})$ with $\|z\| > 2$, then there is some $z_0 \in \bigcup_{n\in\Delta} \mathfrak{A}_n$ with $3K^2\|z - z_0\| < 1$. Also $\mathfrak{A}^{-1} \cap \bigcup_{n\in\Delta} \mathfrak{A}_n \neq \varnothing$, so for some $n_0 \in \Delta$ we have $z_0 \in \mathfrak{A}_{n_0}$ and $\mathfrak{A}_{n_0} \cap \mathfrak{A}^{-1} \neq \varnothing$. Then $\nu_{n_0} : L^1(G_{n_0}) \to \mathfrak{A}$ is a homomorphism with $\operatorname{rng} \nu_{n_0} \cap \mathfrak{A}^{-1} \neq \varnothing$, so by Lemma 2.1.1, we have $e \in \mathfrak{A}_{n_0}$. Replace Δ by $\{\, n \in \Delta : n \ge n_0 \,\}$, so that we can assume without loss that $\{z_0, e\} \subseteq \mathfrak{A}_n$ for each $n \in \Delta$.

As in the proof of Theorem 2.2.1, we have $\Psi_n : \mathfrak{A}_n \to C_b(G_n)$ given by $\Psi_n(a)(x) = \psi(\tilde{\nu}_n(\delta_{x^{-1}})a\tilde{\nu}_n(\delta_x))$ $(a \in \mathfrak{A}_n, x \in G_n)$. Then $\|\Psi_n\| < K^2$. Let M_n be a left-invariant mean on $C_b(G)$, then $M_n \circ \Psi_n \in \mathfrak{A}_n^*$ has $\|M_n \circ \Psi_n\| < K^2$, and $M_n \circ \Psi_n(\tilde{\nu}(\delta_{x^{-1}})a\tilde{\nu}(\delta_x) - a) = 0$, for each $a \in \mathfrak{A}_n$ and $x \in G_n$. Let $f_n \in \mathfrak{A}^*$ be an extension of $M_n \circ \Psi_n$ by the Hahn-Banach Theorem, so that $\|f_n\| = \|M_n \circ \Psi_n\| < K^2$.

Thus we have a bounded net $\{f_n\}_{n\in\Delta}$ in $\mathfrak{A}^*$. By Alaoglu's Theorem, $\{f_n\}_{n\in\Delta}$ has a weak* convergent subnet. We can assume this to be $\{f_n\}_{n\in\Delta}$. Let $f \in \mathfrak{A}^*$ be the weak* limit of this net, then $\|f\| \le K^2$.

Now, if $a, b \in \bigcup_{n\in\Delta} \mathfrak{A}_n$, say $a, b \in \mathfrak{A}_{n_0}$, then $n > n_0 \implies f_n(ab - ba) = 0$, and so $f(ab - ba) = \lim_{n\in\Delta} f_n(ab - ba) = 0$. Hence, as in Theorem 2.2.1, we have that $\overline{\operatorname{span}}\{ab - ba : a, b \in \mathfrak{A}\} \subseteq \ker f$.

Also, for each $n \in \Delta$, $\Psi_n(z_0)(x) - 1 = \psi(\tilde{\nu}(\delta_{x^{-1}})(z_0 - z)\tilde{\nu}(\delta_x))$ $(x \in G_n)$, so $\|\Psi_n(z_0)(x) - 1\| \le 1/3$. Thus $|\tilde{f}_n(z_0) - 1| = |M_n(\Psi_n(z_0) - 1)| < 1/3$. Hence $|f(z_0) - 1| \le 1/3$, and since $|f(z) - f(z_0)| \le 1/3$, we have $f(z) \neq 0$. ∎

So it seems that property $(\mathbf{G}^\infty)$ is not much more helpful in characterizing amenability than properties $(\mathbf{G})$ or $(\mathbf{G}')$.

One last hope for this sort of construction is that we may be able to replace the group algebra occurring as the domain of each homomorphism by an amenable closed $\mathcal{I}_n$ ideal of a group algebra $L^1(G_n)$. Unfortunately, this also fails for the Cuntz algebras, and for $\ell^1_{\mathrm{sym}}(\mathbb{Z})$, by virtue of Lemma 2.1.1.

Chapter 4. Dense-Ranged Homomorphisms

We have seen generalizations of Theorem 1.5.6 to the situation of homomorphisms between the Fourier algebras on piecewise affine sets. In this chapter, we will examine the range of homomorphisms between other types of Banach algebras. In particular, we consider conditions that ensure that a dense-ranged homomorphism is onto.

4.1. Dense-ranged Homomorphisms into Commutative Group Algebras

An easy consequence of Theorem 1.5.6, is that a dense-ranged homomorphism between commutative group algebras is onto. In this section we consider the question of whether the same can be said if we replace one of the algebras by a more general Banach algebra.

Note that in the case where the domain is a commutative group algebra $L^1(G)$, there is an easy answer—if G is finite, then any dense-ranged homomorphism $\ell^1(G) \to \mathfrak{A}$ is onto, whereas if G is infinite, the Fourier transform is a continuous homomorphism $L^1(G) \to C_0(\Gamma)$ with proper dense range. (See [37, Theorems 1.2.4 and 4.6.8].) The case where the codomain is a commutative group algebra has a similar answer.

4.1.1. Proposition. *Suppose G is an infinite locally compact Abelian group. Then there exists a commutative semisimple Banach algebra $\mathfrak{A}$ and a continuous monomorphism $\nu : \mathfrak{A} \to L^1(G)$ with proper dense range.*

Proof. Note that we are seeking a proper dense subalgebra of $L^1(G)$ with a complete algebra norm that dominates a multiple of $\|\cdot\|_1$. It is clear that such a subalgebra is necessarily commutative and semisimple. We consider two cases. If G

is non-discrete, put $\mathfrak{A} = L^1(G) \cap L^2(G)$ with $\|f\| = \|f\|_1 + \|f\|_2$. Then $\langle \mathfrak{A}, \|\cdot\| \rangle$ is a Segal algebra (see [32, section 6.2]) that is a proper dense subalgebra of $L^1(G)$.

If G is discrete we look for a weight function $\omega : G \to [1, \infty)$ that is submultiplicative and unbounded, since then $\ell^1(G, \omega) = \{f \in \ell^1(G) : f \cdot \omega \in \ell^1(G)\}$ is a Beurling algebra (see [32, section 6.3]) that is a proper dense subalgebra of $\ell^1(G)$.

Supposing H is a subgroup of G and that we have such a weight ω' on G/H, then $\omega = \omega' \circ Q_H$ is an appropriate weight on G. By [37, Theorem 2.5.2], every Abelian group has a countably infinite quotient, so we can assume that G is countable.

Take a sequence $\{e = x_1, x_2, \ldots\} \subseteq G$ that generates G and such that for each k, $x_{k+1} \notin G_k$, the subgroup generated by $\{x_1, \ldots, x_k\}$. If this sequence is finite, say $\{e\} = G_1 \subset G_2 \subset \cdots \subset G_n = G$, then some G_{k+1}/G_k is infinite, and being singly generated, must be isomorphic to $\mathbb{Z}$. Let K be the maximum such k, then $G/G_K \cong \mathbb{Z} \times F$ for F a finite group, and since $\mathbb{Z} \times F$ can be given the unbounded submultiplicative weight $\omega(n, x) = 1 + |n|$, we are done.

Otherwise, $G_1 \subset G_2 \subset \cdots$ is infinite, in which case we define an unbounded submultiplicative weight on G by $\omega(x) = \min\{n \geq 1 : x \in G_n\}$. ∎

The proper dense subalgebras of $L^1(G)$ considered in Proposition 4.1.1 are not amenable. If G is non-discrete, the Segal algebra $\mathfrak{A} = L^1(G) \cap L^2(G)$ has $\mathfrak{A} * \mathfrak{A} \subseteq L^2(G) * L^2(G) \subseteq A(G) \subset C_0(G)$. Let $K \subseteq G$ be a compact non-open set, then $\chi_K \in \mathfrak{A}$, but $\chi_K \notin C_0(G)$, so χ_K does not factor. Hence $\mathfrak{A}$ does not factor, and cannot be amenable. More generally, if $\mathfrak{A}$ is a proper Segal algebra in $L^1(G)$, for G *any* locally compact group, then by [9, Theorem 1.2], $\mathfrak{A}$ does not have bounded approximate identity, so that $\mathfrak{A}$ is not amenable.

Next, if G is a locally compact group and $\omega : G \to (0, \infty)$ is a submultiplicative continuous weight on G, consider $\mathfrak{A} = L^1(G, \omega)$. Clearly $\mathfrak{A} \subseteq L^1(G)$ if and only if ω is bounded below. Also, by [18, Theorem 0], $\mathfrak{A}$ is amenable if and only if G is amenable and $\{\omega(x)\omega(x^{-1}) : x \in G\}$ is bounded. Hence, if $\mathfrak{A}$ is an amenable Beurling subalgebra of $L^1(G)$, then $\{\omega(x) : x \in G\}$ is bounded, so that $\mathfrak{A} = L^1(G)$.

It is interesting to compare this with [40], where it is shown that for a weight on an amenable group G, there is a homomorphism $\psi : G \to \mathbb{R}^+$ such that $\omega(x) \geq \psi(x)$ $(x \in G)$, and so the weight $\omega'(x) = \omega(x)/\psi(x)$ is bounded below by 1. Then the mapping $f \mapsto f \cdot \psi$ is an isometric isomorphism $L^1(G, \omega) \to L^1(G, \omega')$. Thus any Beurling algebra on an amenable locally compact group G is isometric to a Beurling subalgebra of $L^1(G)$.

These observations about the non-amenability of certain proper dense subalgebras of group algebras raise the possibility that if $\mathfrak{A}$ is an amenable Banach algebra and $\nu : \mathfrak{A} \to L^1(G)$ is a continuous homomorphism with dense range, then ν is onto. We investigate this in the next section.

4.2. Minimality

4.2.1. Definitions. Suppose $\mathfrak{B}$ is a Banach algebra. A *Banach subalgebra* of $\mathfrak{B}$ is a subalgebra $\mathfrak{A}$ with its own complete algebra norm $\|\cdot\|_{\mathfrak{A}}$ such that the inclusion mapping $\mathfrak{A} \hookrightarrow \mathfrak{B}$ is continuous. We say $\mathfrak{B}$ is *minimal* if it has no proper dense Banach subalgebra. We say $\mathfrak{B}$ is *minimal-amenable* if it is amenable, but has no proper dense amenable Banach subalgebra. More generally, if **(P)** is any property of Banach algebras, we say that a Banach algebra $\mathfrak{A}$ is **(P)**-minimal if $\mathfrak{A}$ has **(P)** and $\mathfrak{A}$ has no proper dense Banach subalgebra with **(P)**.

Note that the continuity criterion is sometimes superfluous—if $\mathfrak{B}$ is commutative and semisimple, then any homomorphism $\mathfrak{A} \to \mathfrak{B}$ is automatically continuous, and so the inclusion $\mathfrak{A} \hookrightarrow \mathfrak{B}$ is continuous. We could make the above definitions with the continuity criterion omitted, but it seems more natural to consider continuous homomorphisms, due to Proposition 0.2.4.

In terms of minimality, some results we have already seen are that for G an infinite locally compact Abelian group, $L^1(G)$ is not minimal and $C_0(G)$ is not minimal-amenable. However, $L^1(G)$ is (G)-minimal. In fact, we have the following.

4.2.2. Proposition. *Suppose G is a locally compact Abelian group and $\mathcal{I}$ is an ideal of $L^1(G)$ with bounded approximate identity. Then $L^1(G)/\mathcal{I}$ is* **(G)***-minimal.*

Proof. By [28, Theorem 13], $\mathcal{I} = \mathcal{I}(X)$, for some $X \in \mathcal{R}_c(\Gamma)$, and then $L^1(G)/\mathcal{I} \cong A(X)$. Thus if $\mathfrak{A}$ is a proper dense subalgebra of $L^1(G)/\mathcal{I}$ with property **(G)**, we obtain a locally compact Abelian group G' and a homomorphism $\nu : L^1(G') \to A(X)$ with range dense in $A(X)$. Then, by Theorem 1.6.9, $\operatorname{rng}\nu = \kappa(\alpha)$, which is closed, and so ν is onto. Thus $\mathfrak{A} = L^1(G)/\mathcal{I}$ and so $L^1(G)/\mathcal{I}$ is **(G)**-minimal. ∎

We develop some basic properties of such minimality conditions.

4.2.3. Proposition. *Suppose* **(P)** *is a property of Banach algebras that is preserved by taking the quotient by a closed ideal. Then a Banach algebra $\mathfrak{B}$ with* **(P)** *is* **(P)***-minimal if and only if any dense-ranged homomorphism from a Banach algebra with* **(P)** *into $\mathfrak{B}$ is onto.*

Proof. Suppose $\mathfrak{B}$ is a **(P)**-minimal Banach algebra, $\mathfrak{A}$ is a Banach algebra with **(P)**, and $\nu : \mathfrak{A} \to \mathfrak{B}$ is a dense-ranged homomorphism. Put $\mathcal{I} = \ker\nu$, $\mathfrak{A}' = \operatorname{rng}\nu$ and norm $\mathfrak{A}'$ by $\|\nu(a)\|_{\mathfrak{A}'} = \inf_{z\in\mathcal{I}}\|a+z\|_{\mathfrak{A}} = \|a+\mathcal{I}\|_{\mathfrak{A}/\mathcal{I}}$. Then $\mathfrak{A}'$ is a dense Banach subalgebra of $\mathfrak{B}$. Also, $\mathfrak{A}' \cong \mathfrak{A}/\mathcal{I}$ has **(P)**, so that $\mathfrak{A} = \mathfrak{A}'$ and ν is onto. The converse is trivial. ∎

Suppose **(P)** is a property of Banach algebras. Define **(P$^\sharp$)** to be the property given by "$\mathfrak{A}$ has **(P$^\sharp$)** if and only if $\mathfrak{A}$ is unital and $\mathfrak{A}$ has **(P)**."

4.2.4. Proposition. *Suppose* **(P)** *is a property of Banach algebras preserved by adjoining a unit. Then a unital Banach algebra $\mathfrak{A}$ is* **(P)***-minimal if and only if $\mathfrak{A}$ is* **(P$^\sharp$)***-minimal.*

Proof. It is clearly enough to check that if $\mathfrak{A}$ is **(P$^\sharp$)**-minimal and $\mathfrak{B}$ is a dense Banach subalgebra of $\mathfrak{A}$ with **(P)**, then $e \in \mathfrak{B}$. Supposing $e \notin \mathfrak{B}$, then $\mathfrak{B} + \mathbb{C}e$ is a dense unital Banach subalgebra with **(P$^\sharp$)**, and so $\mathfrak{B}+\mathbb{C}e = \mathfrak{A}$. Then $\mathfrak{B}$ is a maximal ideal of the unital algebra $\mathfrak{A}$, so that $\mathfrak{B}$ is closed, and not dense. (Contradiction.) ∎

4.2.5. Proposition. *Suppose* $(\mathbf{P})$ *is a property of Banach algebras such that a non-unital Banach algebra* $\mathfrak{A}$ *has* $(\mathbf{P})$ *if and only if* $\mathfrak{A}^\sharp$ *has* $(\mathbf{P})$*. Then a non-unital Banach algebra* $\mathfrak{A}$ *is* $(\mathbf{P})$*-minimal if and only if* $\mathfrak{A}^\sharp$ *is* $(\mathbf{P})$*-minimal.*

Proof. Suppose $\mathfrak{A}$ is $(\mathbf{P})$-minimal, and $\mathfrak{B}$ is a dense Banach subalgebra of $\mathfrak{A}^\sharp$ with $(\mathbf{P}^\sharp)$. Clearly the unit elements of $\mathfrak{A}^\sharp$ and $\mathfrak{B}$ coincide. Let $\varphi_0 : \mathfrak{A}^\sharp \to \mathbb{C}$ be the homomorphism $(a, z) \mapsto z$. Then $\varphi_0|_{\mathfrak{B}} \in \Phi_{\mathfrak{B}}$, so $\mathfrak{B}_0 = \mathfrak{A} \cap \mathfrak{B}$ is a maximal ideal of $\mathfrak{B}$ of codimension 1. Hence $\mathfrak{B}_0 + \mathbb{C}e = \mathfrak{B}$. Giving $\mathfrak{B}_0$ the norm from $\mathfrak{B}$, $\mathfrak{B}_0$ has $(\mathbf{P})$, and is a dense Banach subalgebra of $\mathfrak{A}$, so that $\mathfrak{B}_0 = \mathfrak{A}$. Thus $\mathfrak{B} = \mathfrak{A}^\sharp$, and so $\mathfrak{A}^\sharp$ is $(\mathbf{P}^\sharp)$-minimal. Then by Proposition 4.2.4, $\mathfrak{A}^\sharp$ is $(\mathbf{P}^\sharp)$-minimal.

Conversely, if $\mathfrak{A}^\sharp$ is $(\mathbf{P})$-minimal and $\mathfrak{B}$ is a dense Banach subalgebra of $\mathfrak{A}$ with $(\mathbf{P})$, then $\mathfrak{B} + \mathbb{C}e$ is a dense Banach subalgebra of $\mathfrak{A}^\sharp$ with $(\mathbf{P}^\sharp)$. Hence $\mathfrak{B} + \mathbb{C}e = \mathfrak{A}^\sharp$, and $\mathfrak{B} = \mathfrak{A}$. ∎

The above propositions clearly apply to minimality. Also, since quotients, unitizations and finite-codimensional ideals of amenable Banach algebras are amenable (by Proposition 0.2.4, [23, Proposition 5.1], and [13, Corollary 3.8], respectively), the above propositions apply to minimal-amenability. The following lemma is more specific to these two cases.

4.2.6. Lemma. *If* $\mathfrak{A}$ *is a minimal (respectively minimal-amenable) Banach algebra and* $\mathcal{I}$ *is a closed ideal (respectively amenable closed ideal), then* $\mathfrak{A}/\mathcal{I}$ *is a minimal (respectively minimal-amenable) Banach algebra.*

Proof. Consider first the minimal case. Suppose $\nu : \mathfrak{B} \to \mathfrak{A}/\mathcal{I}$ is a monomorphism with range dense in $\mathfrak{A}/\mathcal{I}$. With $Q : \mathfrak{A} \to \mathfrak{A}/\mathcal{I}$ the quotient mapping, put $\mathfrak{A}' = Q^{-1}(\nu(\mathfrak{B}))$, and since ν is 1-1, we can define $p : \mathfrak{A}' \to [0, \infty)$ by $p(a) = \|\nu^{-1}(a + \mathcal{I})\|_{\mathfrak{B}}$. Then p is a submultiplicative seminorm on $\mathfrak{A}'$, so that

$\|a\|_{\mathfrak{A}'} = \|a\|_{\mathfrak{A}} + p(a)$ defines an algebra norm on $\mathfrak{A}'$. Consider $\mathfrak{A}'/\mathcal{I}$. For each $a \in \mathfrak{A}'$, put $b = \nu^{-1}(a+\mathcal{I}) \in \mathfrak{B}$. Then

$$\begin{aligned}\|b\|_{\mathfrak{B}} = p(a) \leq \|a+\mathcal{I}\|_{\mathfrak{A}'/\mathcal{I}} &\leq \inf_{x\in\mathcal{I}}\|a+x\|_{\mathfrak{A}} + \inf_{x\in\mathcal{I}} p(a+x)\\ &= \|a+\mathcal{I}\|_{\mathfrak{A}/\mathcal{I}} + \|b\|_{\mathfrak{B}}\\ &\leq (\|\nu\|+1)\|b\|_{\mathfrak{B}}.\end{aligned}$$

Hence $\mathfrak{A}'/\mathcal{I} \cong \mathfrak{B}$. Then $\mathfrak{A}'/\mathcal{I}$ and $\mathcal{I}$ are complete, so $\mathfrak{A}'$ is a Banach algebra. Also, we have that the inclusion mapping $\mathfrak{A}' \to \mathfrak{A}$ is continuous and $\mathfrak{A}'$ is dense in $\mathfrak{A}$, so that by minimality, $\mathfrak{A}' = \mathfrak{A}$ and $\nu(\mathfrak{B}) = \mathfrak{A}/\mathcal{I}$. Thus $\mathfrak{A}/\mathcal{I}$ is minimal.

In the case where $\mathfrak{A}$ is minimal-amenable and $\mathcal{I}$ is amenable, we consider a dense-ranged monomorphism $\nu : \mathfrak{B} \to \mathfrak{A}/\mathcal{I}$, where $\mathfrak{B}$ is amenable. Then $\mathfrak{A}'/\mathcal{I} \cong \mathfrak{B}$ and $\mathcal{I}$ are both amenable, so that by [23, Proposition 5.1], $\mathfrak{A}'$ is amenable, and we can again conclude that $\mathfrak{A}' = \mathfrak{A}$ and $\nu(\mathfrak{B}) = \mathfrak{A}/\mathcal{I}$. ∎

It is not clear whether the amenability of $\mathcal{I}$ is necessary for the "minimal-amenable" version of the above. Nor is it clear whether commutative group algebras are minimal-amenable. We do, however, have the following, which should be contrasted with Proposition 4.2.2.

4.2.7. Proposition. *Suppose G is an infinite locally compact Abelian group. Then $L^1(G)$ has an ideal $\mathcal{I}$ such that $L^1(G)/\mathcal{I}$ is not* (G)*-minimal, and hence not minimal-amenable.*

Proof. If G is compact, then by [37, Theorem 5.7.5], there is an infinite Sidon set E contained in Γ. Then $L^1(G)/\mathcal{I}(E) \cong c_0(E)$. Let Γ' be a discrete group of the same cardinality as E, so that $c_0(E) \cong c_0(\Gamma')$. However, $A(\Gamma')$ is a proper dense Banach subalgebra of $c_0(\Gamma')$, and hence $L^1(G)/\mathcal{I}(E)$ is not (G)-minimal.

If, on the other hand, G is not compact, then by [37, Theorems 5.2.2 and 5.6.6], there exists a compact Helson set $E \subseteq \Gamma$ homeomorphic to the Cantor set. Then $L^1(G)/\mathcal{I}(E) \cong C(E) \cong C(\prod_1^\infty \mathbb{Z}_2)$, which is again not (G)-minimal, as $A(\prod_1^\infty \mathbb{Z}_2)$ is a proper dense subalgebra of $C(\prod_1^\infty \mathbb{Z}_2)$. ∎

We should note that the term "minimal" (or "minimal-amenable", etc) is only supposed to indicate the lack of a certain type of dense subalgebra, and as such, only refers to an ordering (by inclusion) of such dense subalgebras. It is tempting to lift this to an order on the category of Banach algebras (or the category of amenable Banach algebras, etc). Such an order would be defined by $\mathfrak{A} \preceq \mathfrak{B}$ if there is a dense-ranged monomorphism $\mathfrak{A} \to \mathfrak{B}$. However, it is possible to have non-isomorphic Banach algebras $\mathfrak{A}$, $\mathfrak{B}$ with $\mathfrak{A} \preceq \mathfrak{B} \preceq \mathfrak{A}$. We give an example where both $\mathfrak{A}$ and $\mathfrak{B}$ have property (G). Put $\mathfrak{A} = \mathfrak{A}_1 \oplus \mathfrak{A}_2 \oplus \mathfrak{A}_3$, and $\mathfrak{B} = \mathfrak{B}_1 \oplus \mathfrak{B}_2$, where $\mathfrak{A}_1 = \mathfrak{B}_1 = A(\mathbb{Z} \times \mathbb{R})$, $\mathfrak{A}_3 = \mathfrak{B}_2 = C_0(\mathbb{Z} \times \mathbb{R})$, and $\mathfrak{A}_2 = \{f \in C_0(\mathbb{Z} \times \mathbb{R}) : f(n, \cdot) \in A(\mathbb{R}) \ (n \in \mathbb{Z})\} \cong \bigoplus_{\mathbb{Z}}{}_0\, A(\mathbb{R})$. Then $\mathfrak{B}_1 \oplus \mathfrak{B}_1 \cong \mathfrak{B}_1$, $\mathfrak{B}_2 \oplus \mathfrak{B}_2 \cong \mathfrak{B}_2$ and $\mathfrak{B}_1 \preceq \mathfrak{A}_2 \preceq \mathfrak{B}_2$. Hence

$$\mathfrak{B} \cong \mathfrak{B}_1 \oplus \mathfrak{B}_1 \oplus \mathfrak{B}_2 \preceq \mathfrak{A} \preceq \mathfrak{B}_1 \oplus \mathfrak{B}_2 \oplus \mathfrak{B}_2 \cong \mathfrak{B}.$$

Suppose $\mathfrak{A} \cong \mathfrak{B}$, so that there is an isomorphism $\nu : \mathfrak{A} \to \mathfrak{B}$. Now, each of $\mathfrak{A}_1, \mathfrak{A}_2, \mathfrak{A}_3, \mathfrak{B}_1, \mathfrak{B}_2$ has carrier space $\mathbb{Z} \times \mathbb{R}$, and so $\nu^*|_{\Phi_{\mathfrak{B}}}$ is a homeomorphism

$$\alpha : (\mathbb{Z} \times \mathbb{R}) \cup (\mathbb{Z} \times \mathbb{R}) \to (\mathbb{Z} \times \mathbb{R}) \cup (\mathbb{Z} \times \mathbb{R}) \cup (\mathbb{Z} \times \mathbb{R}).$$

Consider a coset $E_1 = \{n\} \times \mathbb{R} \subseteq \Phi_{\mathfrak{B}_1}$, then $\mathfrak{B}|_{E_1} \cong A(\mathbb{R})$, and so $\mathfrak{A}|_{\alpha(E_1)} \cong A(\mathbb{R})$. However, if $\alpha(E_1) \subseteq \Phi_{\mathfrak{A}_3}$, then $\mathfrak{A}|_{\alpha(E_1)} \cong C_0(\mathbb{R})$. Hence $\alpha(E)$ is either one of the lines in $\Phi_{\mathfrak{A}_1}$ or one of the lines in $\Phi_{\mathfrak{A}_2}$. Similarly, if $E_2 = \{m\} \times \mathbb{R} \subseteq \Phi_{\mathfrak{B}_2}$, then $\alpha(E_2) \subseteq \Phi_{\mathfrak{A}_3}$. Hence $\nu(\mathfrak{A}_1 \oplus \mathfrak{A}_2) = \mathfrak{B}_1$ and $\nu(\mathfrak{A}_3) = \mathfrak{B}_2$. For $r = 1, 2$, put $Y_r = \alpha^{-1}(\Phi_{\mathfrak{A}_r}) \subseteq \Phi_{\mathfrak{B}_1}$. Then since the monomorphism $\nu|_{\mathfrak{A}_1} : \mathfrak{A}_1 \to \mathfrak{B}_1$ is a homomorphism of group algebras, $\alpha|_{Y_1}$ is piecewise affine. Thus $Y_1 \in \mathcal{R}(\mathbb{Z} \times \mathbb{R})$ is piecewise-affinely homeomorphic to $\mathbb{Z} \times \mathbb{R}$. By considering the structure of an element of $\mathcal{R}(\mathbb{Z} \times \mathbb{R})$, it is easily shown that $Y_2 = (\mathbb{Z} \times \mathbb{R}) \setminus Y_1 \in \mathcal{R}(\mathbb{Z} \times \mathbb{R})$ is also piecewise-affinely homeomorphic to $\mathbb{Z} \times \mathbb{R}$. Thus $\mathfrak{B}_1|_{Y_2} \cong A(\mathbb{Z} \times \mathbb{R})$, and so $\mathfrak{A}_2 \cong A(\mathbb{Z} \times \mathbb{R})$. This is clearly not the case.

4.3. Minimality in Finitely-Generated Commutative Banach Algebras

Suppose $\mathfrak{A}$ is a commutative unital Banach algebra, generated as a unital Banach algebra by elements $a_1, \ldots, a_n$. Then there is a natural homomorphism ν_0 from $\mathbb{C}[z_1, \ldots, z_n]$, the ring of polynomials in n variables with complex coefficients, into $\mathfrak{A}$, given by $\nu_0(p) = p(a_1, \ldots, p_n)$. Since $a_1, \ldots, a_n$ generate $\mathfrak{A}$, rng ν_0 is dense in $\mathfrak{A}$. Suppose $\|\cdot\|$ is an algebra norm on $\mathbb{C}[z_1, \ldots, z_n]$ such that ν_0 is continuous, and let $\mathfrak{B}$ the completion of $\langle \mathbb{C}[z_1, \ldots, z_n], \|\cdot\| \rangle$. Then ν_0 extends by continuity to a continuous dense-ranged homomorphism $\nu : \mathfrak{B} \to \mathfrak{A}$. We apply this construction, in the form of a functional calculus, to determine when a finitely-generated Banach algebra is minimal.

4.3.1. Definitions. A *polydisc* is a set of the form

$$\Delta = \Delta_r^n = \left\{ z \in \mathbb{C}^n : |z_k| < r_k, \ (1 \le k \le n) \right\}$$

where n is the *dimension* of Δ and $r \in (0, \infty)^n$ is the *polyradius* of Δ. We will use various abbreviations—omitting the superscript n if not required, and using a single value $t \in (0, \infty)$ for the polyradius $(t, \ldots, t)$, so that $\Delta_t = \Delta_{(t, \ldots, t)}$. Now, for such Δ, the polynomials in n complex variables form a subalgebra of $C(\overline{\Delta})$, whose closure (in the uniform topology) we call the *polydisc algebra* on $\overline{\Delta}$, denoted $\mathcal{A}(\overline{\Delta})$.

The algebra $\mathcal{A}(\overline{\Delta})$ is the set of functions $\overline{\Delta} \to \mathbb{C}$ that are holomorphic on Δ and continuous on $\overline{\Delta}$. It is a commutative semisimple Banach algebra whose carrier space can be naturally identified with $\overline{\Delta}$. It is not a regular Banach algebra, a fact that makes our subsequent investigations more difficult.

Now, the functional calculus of [8, section 20], is not quite suited to our needs—it does not guarantee a homomorphism from an algebra of holomorphic functions into $\mathfrak{A}$ unless $\mathfrak{A}$ is semisimple. For this reason, we introduce a semisimple algebra between the algebra of holomorphic functions and $\mathfrak{A}$. This restricts our functional calculus to functions that are holomorphic on a sufficiently large polydisc. As we will only be considering holomorphic functions on polydiscs, anyway, this does not concern us. In the following, $r(a) = \lim_{n\to\infty}\|a^n\|^{\frac{1}{n}}$ is the *spectral radius* of an element $a \in \mathfrak{A}$, and the *joint spectrum* of $(a_1,\ldots,a_n)$ is $\sigma_{\mathfrak{A}}(a_1,\ldots,a_n) = \{(\varphi(a_1),\ldots,\varphi(a_n)) : \varphi \in \Phi_{\mathfrak{A}}\}$. Clearly if ρ is a polyradius with each $\rho_k \geq r(a_k)$, then $\sigma_{\mathfrak{A}}(a_1,\ldots,a_n) \subseteq \overline{\Delta}_\rho$.

4.3.2. Lemma. *Suppose $a_1,\ldots,a_n$ are commuting elements of a unital Banach algebra $\mathfrak{A}$. Then the natural homomorphism $\nu_0 : \mathbb{C}[z_1,\ldots,z_n] \to \mathfrak{A}$ extends uniquely to a continuous homomorphism $\nu_r : \mathcal{A}(\overline{\Delta}_r) \to \mathfrak{A}$, where r is a polyradius with $r_k > r(a_k)$, for each $1 \leq k \leq n$. Furthermore, if r' is another polyradius with each $r'_k \geq r_k$, and $f \in \mathcal{A}(\overline{\Delta}_{r'})$, then $\nu_{r'}(f) = \nu_r(f|_{\overline{\Delta}_r})$.*

Proof. For each $1 \leq k \leq n$, take $\rho_k > 0$ with $r_k > \rho_k \geq r(a_k)$. Then for each k, $\rho_k^{-n}\|a^n\|$ is convergent to 0 or 1. In either case $\{\rho_k^{-n}\|a^n\|\}_0^\infty$ is bounded, say $\rho_k^{-n}\|a^n\| \leq M_k$, for each n. Put $M = \prod_1^n M_k$. Let S be the semigroup $\mathbb{Z}_+^n$, and define a weight ω on S by $\omega(s_1,\ldots,s_n) = \prod_1^n \rho_k^{s_k}$. Note that $c_{00}(S) \cong \mathbb{C}[z_1,\ldots,z_n]$, and so we have $\psi_0 : c_{00}(S) \to \mathfrak{A}$ given by $\psi_0(\delta_{(s_1,\ldots,s_n)}) = a_1^{s_1}\ldots a_n^{s_n}$, extended by linearity. Giving $c_{00}(S)$ the norm from $\ell^1(S,\omega)$, we have

$$\begin{aligned}
\|\psi_0(f)\| &\leq \sum_{s\in S} |f_s|\,\|\psi_0(\delta_s)\| \\
&\leq \sum_{s\in S} |f_s|\,\|a_1^{s_1}\|\ldots\|a_n^{s_n}\| \\
&\leq \sum_{s\in S} |f_s|\,M_1\rho_1^{s_1}\ldots M_n\rho_n^{s_n} \\
&\leq M\sum_{s\in S} |f_s|\omega(s) \\
&= M\|f_s\|_{\ell^1(S,\omega)}.
\end{aligned}$$

Thus ψ_0 is continuous, and the extension of ψ_0 by continuity is a homomorphism $\psi_\rho : \ell^1(S,\omega) \to \mathfrak{A}$.

It is readily shown that $\ell^1(S,\omega)$ has carrier space $\{\varphi_z : z \in \overline{\Delta}_\rho\}$, where $\varphi_z(f) = \sum_{s\in S} f(s)z_1^{s_1}\ldots z_n^{s_n}$, and the topology on $\Phi_{\ell^1(S,\omega)}$ corresponds to the standard topology on $\overline{\Delta}_\rho$. With this, $\ell^1(S,\omega)$ is semisimple. Moreover, $\ell^1(S,\omega)$

has generators $e_1 = \delta_{(1,\ldots,0)}, \ldots, e_n = \delta_{(0,\ldots,1)}$, whose joint spectrum is $\overline{\Delta}_\rho$. Thus we can apply the functional calculus of [8, Corollary 20.6], so that for any $r > \rho$, there is a continuous homomorphism $\theta_{r,\rho} : \mathcal{A}(\overline{\Delta}_r) \to \ell^1(S,\omega)$ with $\theta_{r,\rho}(z_k) = e_k$. Composing this with ψ_ρ gives a continuous homomorphism $\nu_r : \mathcal{A}(\overline{\Delta}_r) \to \mathfrak{A}$ such that each $\nu_r(z_k) = a_k$, and so ν_r is an extension of ν_0. The uniqueness of ν_r follows from the fact that $\mathbb{C}[z_1, \ldots, z_n]$ is dense in $\mathcal{A}(\overline{\Delta}_r)$.

Finally, if r' is a polyradius with $\Delta_r \subseteq \Delta_{r'}$, then the restriction homomorphism $\rho_{\overline{\Delta}_r} : \mathcal{A}(\overline{\Delta}_{r'}) \to \mathcal{A}(\overline{\Delta}_r)$, given by $f \mapsto f|_{\overline{\Delta}_r}$, is norm-reducing, and consequently continuous. Then $\rho_{\overline{\Delta}_r}(p) = p$, for any $p \in \mathbb{C}[z_1, \ldots, z_n]$, and so $\nu_r \circ \rho_{\overline{\Delta}_r}$ is an extension of ν_0 to a continuous homomorphism $\mathcal{A}(\overline{\Delta}_{r'}) \to \mathfrak{A}$. Hence $\nu_r \circ \rho_{\overline{\Delta}_r} = \nu_r$. ∎

An alternative approach to the homomorphism $\theta_{r,\rho}$ above that does not use the existing functional calculus is as follows. We can consider $\ell^1(S,\omega)$ to be a set of power series in $z_1, \ldots, z_n$. It consists of all holomorphic f whose power series (at 0) is absolutely convergent throughout $\overline{\Delta}_\rho$. It is an elementary theorem of complex analysis that if f is a holomorphic function on a polydisc Δ_r then the power series of f at 0 is absolutely convergent throughout Δ_r. Hence if $\overline{\Delta}_\rho \subseteq \Delta_r$, then the power series at 0 belongs to $\ell^1(S,\omega)$. This gives us a homomorphism $\mathcal{A}(\overline{\Delta}_r) \to \ell^1(S,\omega)$.

A point worth noting is that the homomorphism $\theta_{r,\rho}$ is 1-1, as is any $\rho_{\overline{\Delta}_r}$. This follows by the identity theorem for holomorphic functions. Indeed, for this reason we will consider $\mathcal{A}(\overline{\Delta}_{r'}) \subseteq \mathcal{A}(\overline{\Delta}_r)$ when $\overline{\Delta}_r \subseteq \overline{\Delta}_{r'}$.

Now suppose $\mathfrak{A}$ is minimal and generated by $\{a_1, \ldots, a_n\}$. Then with r as in Lemma 4.3.2, $\nu_r : \mathcal{A}(\overline{\Delta}_r) \to \mathfrak{A}$ is a dense-ranged homomorphism, so by Lemma 4.2.3, ν_r is onto. Thus $\mathfrak{A} \cong \mathcal{A}(\overline{\Delta}_r)/\mathcal{I}_r$, where $\mathcal{I}_r = \ker \nu_r$. Moreover, this occurs for any polyradius r such that each $r_k > r(a_k)$. This seems quite exceptional, for if $\Delta_r \subset \Delta_{r'}$, then the restriction homomorphism $\rho_{\overline{\Delta}_r} : \mathcal{A}(\overline{\Delta}_{r'}) \to \mathcal{A}(\overline{\Delta}_r)$ has proper dense range, whereas $Q_{\mathcal{I}_r} \circ \rho_{\overline{\Delta}_r} : \mathcal{A}(\overline{\Delta}_{r'}) \to \mathcal{A}(\overline{\Delta}_r)/\mathcal{I}_r$ is always onto. We will show that this can occur only in the case where the minimality of $\mathfrak{A} \cong \mathcal{A}(\overline{\Delta}_r)/\mathcal{I}_r$ is trivial—that is, when $\mathfrak{A}$ is finite-dimensional.

By Theorem 1.2.1, we have $\alpha_r = \nu_r^*|_{\Phi_{\mathfrak{A}}} : \Phi_{\mathfrak{A}} \to \overline{\Delta}_r$. Note that $\alpha_r(\varphi) = (\varphi(a_1), \ldots, \varphi(a_n)) \in \mathbb{C}^n$, and so $\alpha_r = \alpha_{r'}$, enabling the omission of the subscript from here on. Thus $\alpha(\Phi_{\mathfrak{A}}) \subseteq \overline{\Delta}_\rho \subset \Delta_r$, where ρ and r are as in Lemma 4.3.2.

4.3.3. Proposition. *A finitely-generated commutative minimal Banach algebra has finite carrier space.*

Proof. Let $\mathfrak{A}$ be a finitely-generated commutative minimal Banach algebra. Without loss, $\mathfrak{A}$ is unital and the generators $a_1, \ldots, a_n$ satisfy $\|a_k\| < 1$. By Lemma 4.2.6, $\mathfrak{A}/\mathrm{rad}\,\mathfrak{A}$ is also minimal, and since the carrier space of $\mathfrak{A}/\mathrm{rad}\,\mathfrak{A}$ is naturally identified with $\Phi_{\mathfrak{A}}$, we can assume that $\mathfrak{A}$ is semisimple. Let r be a polyradius with each $r_k \geq 1$, let $\nu_r : \mathcal{A}(\overline{\Delta}_r) \to \mathfrak{A}$ be as given by Lemma 4.3.2, and let $\mathcal{I}_r = \ker \nu_r$. By the above discussion, ν_r is an epimorphism, so that $\mathfrak{A} \cong \mathcal{A}(\overline{\Delta}_r)/\mathcal{I}_r$.

Now, since we are assuming that $\mathfrak{A}$ is semisimple, we have, by Theorem 1.2.1, that with X_r the hull-kernel closure of $\alpha(\Phi_{\mathfrak{A}})$, $\mathcal{I}_r = \mathcal{I}(\alpha(\Phi_{\mathfrak{A}})) = \mathcal{I}(X_r)$. Moreover, each $f \in \mathcal{I}_r$ is holomorphic on Δ_r, so that $X_r \cap \Delta_r$ is an analytic variety in Δ_r. But $\mathfrak{A} \cong \mathcal{A}(\overline{\Delta}_r)/\ker \nu_r = \mathcal{A}(\overline{\Delta}_r)/\mathcal{I}(X_r)$ which, by [8, Proposition 23.5], has carrier space X_r. It follows that α is a homeomorphism from $\Phi_{\mathfrak{A}}$ onto X_r, so that $X_r \subseteq \Delta_r$. Hence $\alpha(\Phi_{\mathfrak{A}}) = X_r \cap \Delta_r$ is a compact analytic variety in Δ_r, which is necessarily finite, by [19, Corollary III.B.17]. Finally, α is injective, so $\Phi_{\mathfrak{A}}$ is finite. ∎

It is in the following lemma that we use the functional calculus in the non-semisimple case to obtain homomorphisms into a commutative unital Banach algebra $\mathfrak{A}$ that is *local* (that is, $\mathfrak{A}$ has a single maximal ideal). We make use of the local theory of holomorphic functions of several complex variables. We define two holomorphic functions f, g to be *equivalent at 0* if there is a neighbourhood U of zero with $f|_U = g|_U$. We define two analytic varieties V, W to be *equivalent at 0* if there is a neighbourhood U of zero with $V \cap U = W \cap U$. The equivalence classes thus formed we will call the *germs* of holomorphic functions and the *germs* of analytic varieties, respectively. As in [19], we denote the ring of germs of holomorphic functions in n complex variables by ${}_n\mathcal{O}$. In the notation above, this is the union (actually an inductive limit) of the Banach algebras $\mathcal{A}(\overline{\Delta}_r)$ over all polyradii r. (We have seen that these algebras get larger as the polydisc gets smaller.) If $\mathcal{I}$ is an ideal of ${}_n\mathcal{O}$, then $\operatorname{loc}\mathcal{I}$, the *locus* of $\mathcal{I}$, is defined to be the germ of the analytic variety on which all $f \in \mathcal{I}$ are zero. (This is well-defined, by [19, Proposition 2.E.9].) In our notation, this is the germ of a variety determined by the analytic sets $Z(\mathcal{I} \cap \mathcal{A}(\overline{\Delta}_r))$, as Δ_r decreases. We also have the concept of $\operatorname{id}(V)$, the *ideal* of V, where V is a germ of an analytic variety. This is $\operatorname{id}(V)$, the set of $f \in {}_n\mathcal{O}$ such that f is zero on some representative of the germ. This corresponds to the notion of the kernel, and in fact, $\operatorname{id}(V) = \bigcup_r \mathcal{I}_{\mathcal{A}(\overline{\Delta}_r)}(V_r)$, where the union is taken over all polyradii r such that there is a representative V_r of V that is an analytic variety in Δ_r. The most significant theorem of this local theory is [19, Theorems II.E.20 and III.A.7], commonly called the *Nullstellensatz*, which states that for an ideal of ${}_n\mathcal{O}$, we have

$$\operatorname{id}\operatorname{loc}(\mathcal{I}) = \operatorname{rad}\mathcal{I} = \{\, f \in {}_n\mathcal{O} : f^m \in \mathcal{I}, \text{ for some } m \in \mathbb{N} \,\}.$$

4.3.4. Lemma. *A commutative unital Banach algebra that is finitely-generated, local, and minimal is finite dimensional.*

Proof. Suppose $\mathfrak{A}$ is a finitely-generated commutative unital local minimal Banach algebra. Let φ be the unique non-zero homomorphism $\mathfrak{A} \to \mathbb{C}$. If we replace the set of generators $\{\, a_k : 1 \le k \le n \,\}$ by $\{\, a_k - \varphi(a_k)e : 1 \le k \le n \,\}$, we can assume that $\alpha(\Phi_{\mathfrak{A}}) = 0 \in \mathbb{C}^n$. Thus each $r(a_k) = 0$.

Now, for any polydisc $\Delta_r \subseteq \mathbb{C}^n$, we have an epimorphism $\nu_r : \mathcal{A}(\overline{\Delta}_r) \to \mathfrak{A}$, and since $\nu_r(f) = \nu_{r'}(f|_{\Delta_{r'}})$ whenever $\Delta_{r'} \subseteq \Delta_r$, we have a homomorphism $\nu' : {}_n\mathcal{O} \to \mathfrak{A}$. Clearly $\ker\nu' = \bigcup_r \ker\nu_r$, and so $\operatorname{loc}(\ker\nu') = \bigcap_r \operatorname{loc}(\ker\nu_r) = \bigcap_r Z(\ker\nu_r) = \bigcap_r \{0\}$, by Theorem 1.2.1.

Hence, $\mathcal{I} = \ker\nu'$ is an ideal of ${}_n\mathcal{O}$ with $\operatorname{loc}(\mathcal{I}) = \{0\}$, and so by the *Nullstellensatz*, we have that if $f \in {}_n\mathcal{O}$ and $f(0) = 0$, then $f^m \in \mathcal{I}$, for some $m > 0$. In particular, for each $1 \le k \le n$ there exists $m_k > 0$ such that $z_k^{m_k} \in \mathcal{I}$. Thus $a_k^{m_k} = 0$ for each $1 \le k \le n$, and so $\mathfrak{A} = \operatorname{span}\{\, \prod_1^n a_k^{p_k} : \text{each } 0 \le p_k \le m_k \,\}$, which is clearly finite-dimensional. ∎

4.3.5. Theorem. *Suppose $\mathfrak{A}$ is a finitely-generated commutative minimal Banach algebra. Then $\mathfrak{A}$ is finite dimensional.*

Proof. By Proposition 4.2.5, we can suppose, without loss, that $\mathfrak{A}$ is unital. Then by Proposition 4.3.3, $\Phi_{\mathfrak{A}}$ is finite, say $\Phi_{\mathfrak{A}} = \{\varphi_1, \ldots, \varphi_n\}$. Let $a_1, \ldots, a_n \in \mathfrak{A}$ be such that for each $1 \leq k \leq n$, $\hat{a}_k(\varphi_k) = 1$ and for $j \neq k$, $\hat{a}_k(\varphi_j) = 0$. Each a_k has $a_k^2 - a_k \in \operatorname{Rad}\mathfrak{A}$, and so by [31, Theorem 2.3.9], there exists an idempotent $e_k \in \mathfrak{A}$ with $e_k - a_k \in \operatorname{Rad}\mathfrak{A}$, so that $\hat{e}_k = \hat{a}_k$. Now, for $j \neq k$, $e_j e_k$ is an idempotent in $\operatorname{Rad}\mathfrak{A}$ and so $e_j e_k = 0$. Similarly $e - \sum_1^n e_k = 0$. Hence $\mathfrak{A} = \mathfrak{A}e_1 \oplus \cdots \oplus \mathfrak{A}e_n$, a direct sum of principal ideals. For each $1 \leq k \leq n$, put $\mathcal{I}_k = \mathfrak{A}(e - e_k)$, an ideal such that $\mathfrak{A}e_k \cong \mathfrak{A}/\mathcal{I}_k$, so that $\Phi_{\mathfrak{A}e_k}$ is naturally identified with $Z(\mathcal{I}_k) = \{\varphi_k\}$. Hence $\mathfrak{A}e_k$ is a finitely-generated local commutative unital Banach algebra, which is minimal, by Lemma 4.2.6, and consequently finite-dimensional, by Lemma 4.3.4. Hence $\mathfrak{A} = \bigoplus_1^n \mathfrak{A}e_k$ is finite-dimensional. ■

4.4. Some Minimal Algebras

It may seem that we can always find a proper dense Banach subalgebra of an infinite-dimensional Banach algebra $\mathfrak{A}$. We have seen that this is the case for $L^1(G)$ and $C_0(G)$, where G is an infinite locally compact Abelian group, and for finitely-generated Banach algebras. It is also possible to show that if S is a semigroup, then $\ell^1(S)$ has a proper dense subalgebra, we only need construct an unbounded submultiplicative weight ω on S, and then $\ell^1(S,\omega)$ is a proper dense Banach subalgebra of $\ell^1(S)$. In contrast to this, we show that there are many infinite-dimensional Banach algebras that are minimal. We begin with a special case of [16, Definition 2.1].

4.4.1. Definition. A locally compact topological space X is called an *F-space* if $C(X)$, the algebra of *all* continuous functions $X \to \mathbb{C}$, has the property that every finitely generated ideal is a principal ideal.

Any X discrete set is an F-space, and by [16, Theorem 2.3], any X is an F-space if and only if βX, the Stone-Čech compactification of X, is an F-space. We then have the following result from [6, Theorem A].

4.4.2. Proposition. *If X is a compact F-space, then $C(X)$ is a minimal Banach algebra.* ∎

Note that any homomorphism $\mathfrak{A} \to C(X)$ is automatically continuous, and so the continuity criterion in definition 4.2.1 is not needed. Since the algebras $C(X)$ are amenable—they have property (G)—we have the following.

4.4.3. Corollary. *If X is a compact F-space, then $C(X)$ is a minimal-amenable Banach algebra.* ∎

Particular examples of this are if S is any set with its discrete topology, then βS is a compact F-space, and so $C(\beta S) \cong \ell^\infty(S)$ is a minimal Banach algebra. The smallest example of this is $\ell^\infty = \ell^\infty(\mathbb{N})$, which is non-separable. More generally, note that by [16, Corollary 2.4], an infinite compact F-space X does not satisfy the first axiom of countability at any $x \in X$, that is, there is no countable base to the topology at any point $x \in X$. It follows that $C(X)$ is non-separable. Thus, our only examples of minimal Banach algebras are finite-dimensional or non-separable. We summarize some examples of Banach algebras that we have seen to be non-minimal, or can easily be shown to be non-minimal. These examples include many separable Banach algebras :

(i) $C_0(X)$, where X is an infinite locally compact Abelian group or an infinite closed subset of $\mathbb{R}^n$, or any continuous image of either of these,

(ii) $L^1(G)$, where G is an infinite locally compact group,

(iii) $\ell^1(S)$, where S is an infinite semigroup, (construct an unbounded submultiplicative weight on S)

and many others. No example is known to the author of an infinite-dimensional separable minimal Banach algebra.

4.5. Conjectures and Questions

We conclude this chapter with some conjectures and open questions. The conjectures are distinguished from the open questions solely by certain prejudices of the author.

4.5.1. Conjecture. *Commutative group algebras are minimal-amenable.*

4.5.2. Question. *Is there a separable commutative minimal Banach algebra?*

We should consider the following rather vague question, as an attempt to attain a characterization of amenability along the lines of property (G). A similar question could be asked about commutative amenable Banach algebras.

4.5.3. Question. *Is there a relatively small, easily-defined class $\mathcal{A}$ of amenable Banach algebras such that any amenable Banach algebra has a dense Banach subalgebra isomorphic to a member of $\mathcal{A}$?*

Evidently such a class will contain all minimal-amenable Banach algebras.

4.5.4. Conjecture. *There is an amenable Banach algebra $\mathfrak{A}$ such that $\mathfrak{A}$ does not contain a minimal-amenable Banach algebra as a proper dense subalgebra.*

That is, the lattice of dense amenable Banach subalgebras of $\mathfrak{A}$ has no minimal elements. If this conjecture fails, the smallest possible class $\mathcal{A}$ in question 4.5.3 would be the class of minimal-amenable Banach algebras. Thus, providing this is a "relatively small, easily-defined" class of Banach algebras, this question would have an affirmative answer.

Any X discrete set is an F-space, and by [16, Theorem 2.?], any X is an F-space if and only if βX, the Stone-Čech compactification of X, is an F-space. We then [illegible]

4.4.2. Proposition. *If X is a compact F-space, then $C(X)$ is a minimal Banach algebra.* ■

Note that any homomorphism $\mathfrak{B} \to C(X)$ is automatically continuous, and so the continuity criterion in definition [illegible] [illegible] are amenable—they have property (G)—we have the following.

4.4.3. Corollary. *If X is a compact F-space, then $C(X)$ is a minimal amenable Banach algebra.*

Particular examples of this are: if S is any set with its discrete topology then βS is a compact F-space, and so $C(\beta S) \cong \ell^\infty(S)$ is a minimal Banach algebra. The smallest example of this is $\ell^\infty = \ell^\infty(\mathbb{N})$, which is non-separable. More generally, note that by [16, Corollary 2.?], an infinite compact F-space X does not satisfy the first axiom of countability [illegible] there is no countable base to the topology at any point $x \in X$; it follows that $C(X)$ is non-separable. Thus, our only examples of minimal Banach algebras are finite-dimensional or non-separable. We summarize some examples of Banach algebras that we have seen to be non-minimal, [illegible] separable Banach algebras:

(i) $C(X)$, where X is [illegible] infinite closed subset of [illegible]

(ii) $L^1(G)$, where G is an infinite locally compact group;

(iii) $\ell^1(S)$, where S is an infinite semigroup (construct an unbounded submultiplicative weight on S)

and many others. No example is known to the author of an infinite-dimensional separable minimal Banach algebra.

Appendix A. Homomorphisms Into Measure Algebras

Let G_1 and G_2 be locally compact Abelian groups. The results of Cohen, used in Chapter 1 to characterize the range of homomorphisms $L^1(G_1) \to L^1(G_2)$, also concern homomorphisms $\nu : L^1(G_1) \to M(G_2)$, and the extension of such homomorphisms to a homomorphism $\tilde{\nu} : M(G_1) \to M(G_2)$. It is natural to consider whether we can derive results characterizing the range of such homomorphisms ν and $\tilde{\nu}$ in a way analogous to Theorem 1.5.6. The sort of characterization of range we seek is one in which there is an equivalence relation $\sim$ on some $Y \in \mathcal{R}(\Gamma_2)$ such that the range is the subalgebra of $M(G_2)$ of measures whose Fourier-Stieltjes transforms are class functions of this equivalence. For this, define

$$\tilde{\kappa}(\sim) = \{\mu \in M(G_2) : \hat{\mu} = 0 \text{ off } Y \text{ and } \hat{\mu}(\gamma_1) = \hat{\mu}(\gamma_2) \text{ whenever } \gamma_1 \sim \gamma_2\}.$$

When $\sim$ is the equivalence relation determined by a function ψ with domain Y, we will use $\tilde{\kappa}(\psi)$ for $\tilde{\kappa}(\sim)$. Again, we will use additive notation for the group product of all groups in this section.

A.1. Homomorphisms from Group Algebras in to Measure Algebras

Let ν be a homomorphism $L^1(G_1) \to M(G_2)$ with $\operatorname{rng} \nu \not\subseteq L^1(G_2)$. Then $\alpha : Y \to \Gamma_1$ (as in Theorem 1.3.3) is non-proper. We consider the range of ν in such a case.

In the analysis of proper piecewise affine maps in Chapter 1, we saw that such a map $\alpha : Y \to \Gamma_1$ is built up from pieces of the form $\tau_{\gamma_1} \circ \psi \circ Q_\Lambda \circ \tau_{-\gamma_2}|_S$, where $S \in \mathcal{R}_0(\Gamma_2)$, $\gamma_2 \in S$, Q_Λ is the quotient by some compact subgroup $\Lambda \subseteq E_0(S) - \gamma_2$, ψ is a topological group isomorphism from $(E_0(S) - \gamma)/\Lambda$ onto Ξ, a closed subgroup of Γ_1, and $\gamma_1 = \alpha(\gamma_2) \subseteq \Gamma_1$. For a non-proper piecewise affine map, the situation is similar, with two possible differences—the closed subgroup Λ could be non-compact, and ψ may no longer have a continuous inverse. We consider embryonic cases of each of these.

Suppose Γ is a locally compact Abelian group and Λ is a closed, non-compact subgroup. Put $\alpha = Q_\Lambda : \Gamma \to \Gamma/\Lambda$. With $H = \mathrm{Ann}_G \Lambda$, we have algebra homomorphisms $\nu : L^1(H) \to M(G)$ and $\tilde{\nu} : M(H) \to M(G)$. Then by [37, Theorem 2.7.1],

$$\begin{aligned}\tilde{\kappa}(\alpha) &= \{\mu \in M(G) : \hat{\mu} \text{ is constant on cosets of } \Lambda\} \\ &= \{\mu \in M(G) : \mu \text{ is concentrated on } H\},\end{aligned}$$

so that $\tilde{\kappa}(\alpha) \cong M(H)$, and $\operatorname{rng} \tilde{\nu} = \tilde{\kappa}(\alpha)$.

Now consider non-proper behaviour of the other type—where ψ^{-1} is not continuous. An extreme, but simple, example of this is where α is the identity map $\Gamma_d \to \Gamma$. In this case the homomorphisms are the natural injections $\hat{\nu} : A(\Gamma) \to B(\Gamma_d)$ and $\hat{\tilde{\nu}} : B(\Gamma) \to B(\Gamma_d)$. By [37, Theorem 1.9.1], the second of these has range $B(\Gamma) = B(\Gamma_d) \cap C(\Gamma)$, whereas the range of the first is a subset of this. These are each point-separating algebras on Γ_d, so that the only possible equivalence relation such that $\operatorname{rng} \nu \subseteq \tilde{\kappa}(\sim)$ is "=". But $\tilde{\kappa}(=)\hat{} = B(\Gamma_d)$, so that unless Γ is discrete, ν is not onto. But if Γ is discrete, then α is proper.

Thus in order to characterize the range of a homomorphism $L^1(G_1) \to M(G_2)$, or its extension $M(G_1) \to M(G_2)$, some topological conditions will also be necessary on the transforms of the measures. These are not in the spirit of the results we seek.

We can, however, avoid this by specifying that Γ_1 is discrete, for then $\Xi = \operatorname{rng} \psi$ will be a closed subgroup and ψ will have continuous inverse. Indeed, it is not difficult to modify the arguments of Section 1.5 to prove the following.

A.1.1. Theorem. *Suppose G_1 is a compact Abelian group, G_2 is a locally compact Abelian group, $\nu : L^1(G_1) \to M(G_2)$ is an algebra homomorphism and*

$Y \in \mathcal{R}(\Gamma_2)$ and $\alpha : Y \to \Gamma_1$ are as in Theorem 1.3.3. Then the natural extension of ν to a homomorphism $\tilde{\nu} : M(G_1) \to M(G_2)$ has range $\tilde{\kappa}(\alpha)$.

Proof. Since $\alpha(Y) \in \mathcal{R}(\Gamma_1)$, there is a measure $\mu \in M(G_1)$ with $\hat{\mu} = \chi_{\alpha(Y)}$, and it follows that $B(\alpha(Y)) = B(\Gamma_1)|_{\alpha(Y)}$. Thus it suffices to show that for each $\mu \in \tilde{\kappa}(\alpha)$ that $\hat{\mu} \circ \alpha^{-1} \in B(\alpha(Y))$. For this it suffices, by Theorem 1.4.1 to prove the case where $Y \in \mathcal{R}_0(\Gamma_2)$ and α has an affine extension $\alpha_1 : E \to \Gamma_1$, and $E = E_0(Y)$. Let Ξ be the closed subgroup such that $\alpha_1 \circ Q_\Xi^{-1}$ is injective. We can now apply a modified version of the "smudging" technique used in Lemma 1.5.1, using the set $F \subseteq \Xi$ such that $S + (\Lambda_0 \cap \Xi) = S' + F = S + F$ in Proposition 1.4.12 in place of the set F from Lemma 1.4.3. Since $\Xi \subseteq \Lambda_0 = E_0 - E_0$, we have that the smudging technique gives $\hat{\mu}_1 \in \tilde{\kappa}(\alpha_1)$ with $\hat{\mu}_1 \cdot \chi_Y = \hat{\mu}$. We can now apply the case of a quotient by a closed subgroup, analysed above, to obtain $\hat{\mu}_1 \circ \alpha_1^{-1} \in B(\alpha_1(E_0))$, giving finally $\hat{\mu} \circ \alpha^{-1} \in B(\alpha(Y))$. ∎

A.2. Extensions of Group Algebra Homomorphisms

We now investigate homomorphisms $M(G_1) \to M(G_2)$ that arise from *proper* piecewise affine maps $\alpha : Y \to \Gamma_1$. This is considerably more fruitful than the non-proper case.

We start with the case where Y is an open coset in Γ_2 and $\alpha : E \to \Gamma_1$ is affine. In this case we can argue as in Lemma 1.3.4 almost verbatim, starting with $\mu \in \tilde{\kappa}(\alpha)$, and deducing that $\hat{\mu} \circ \alpha^{-1} \in B(\alpha(Y))$. Then $B(\alpha(Y)) = B(\Gamma_1)|_{\alpha(Y)}$, so that there exists $\mu_0 \in M(G_1)$ with $\hat{\mu}_0|_{\alpha(Y)} = \hat{\mu} \circ \alpha^{-1}$, and hence $\nu(\mu_0) = \mu$.

The next case is analogous to Lemma 1.5.1. When $Y \in \mathcal{R}_0(\Gamma_2)$ and $\alpha : Y \to \Gamma_1$ has a proper affine extension $\alpha_1 : E_0(Y) \to \Gamma_1$, we can apply the same "smudging" argument (by a compact subgroup) to $\mu \in \tilde{\kappa}(\alpha)$, giving $\hat{\mu}|_Y = \tilde{F}|_Y$, for some $\tilde{F} \in \tilde{\kappa}(\alpha_1)$. Then $\tilde{F} \circ \alpha_1^{-1} \in B(\Gamma_1)|_{\alpha(E_0(S))}$, and so $\hat{\mu} \circ \alpha^{-1} \in M(\Gamma_1)|_{\alpha(S)}$, as required. It is the final step of combining affine pieces that causes us difficulties.

A.2.1. Example. Let $G_1 = \mathbb{R}$, $G_2 = \mathbb{T}$, so that $\Gamma_1 = \mathbb{R}$ and $\Gamma_2 = \mathbb{Z}$. Take any irrational $\xi \in \mathbb{R}$, and define $\alpha : \mathbb{Z} \to \mathbb{R}$ by $\alpha(2n) = \xi n$ and $\alpha(2n+1) = 2n+1$. Then $\kappa(\alpha) = L^1(\mathbb{T})$, so that the homomorphism $\nu : L^1(\mathbb{R}) \to L^1(\mathbb{T})$ determined by $(\nu(f))\widehat{\ } = \hat{f} \circ \alpha$ is onto. However, ν has a unique extension $\tilde{\nu} : M(\mathbb{R}) \to M(\mathbb{T})$ that is not onto, as we will now show. Let $\mu_2 = {}^1\!/\!{}_2(\delta_1 + \delta_{-1}) \in M(\mathbb{T})$. Clearly $\hat{\mu}_2 = \chi_{2\mathbb{Z}}$, so if $\mu_1 \in M(\mathbb{R})$ had $\tilde{\nu}(\mu_1) = \mu_2$, then the uniform continuity of $\hat{\mu}_1$ would be contradicted by $\hat{\mu}_1(\xi\mathbb{Z}) = \hat{\mu}_1(\alpha(2\mathbb{Z})) = \{1\}$ and $\hat{\mu}_1(2\mathbb{Z}+1) = \hat{\mu}_1(\alpha(2\mathbb{Z}+1)) = \{0\}$.

So we see that for a homomorphism $\nu : L^1(G_1) \to L^1(G_2)$, it may occur that $\operatorname{rng} \tilde{\nu}$ is properly contained within $\tilde{\kappa}(\alpha)$. The difference between this case and that in Theorem 1.5.6 is in Lemma 1.5.3. Some partial results are possible, however. This stems from the observation that the uniform continuity of $\hat{\mu}_1$ played a vital rôle in Example A.2.1.

Recall from [5] that two sets $A, B \subseteq \Gamma$ are called *uniformly separated* if there is a neighbourhood U of $e \in \Gamma$ such that $A + U$ and B are disjoint. We can then call a finite family $A_1, \ldots, A_n$ of closed sets uniformly separated if they are pairwise uniformly separated. Clearly this holds if and only if each A_k is uniformly separated from $\bigcup_{j \neq k} A_j$. In the following theorem, we use the equivalence of uniform separation of sets $A, B \in \mathcal{R}_c(\Gamma)$ and the existence of a separating measure for A, B; that is, a measure $\mu \in M(G)$ such that $\hat{\mu}$ takes the value 0 on A and the value 1 on B. The existence of such μ is dealt with in [5].

A.2.2. Theorem. *Suppose* $Y \in \mathcal{R}(\Gamma_2)$ *and* $\alpha : L^1(G_1) \to L^1(G_2)$ *is a proper piecewise affine map, and for some* $S_1, \ldots, S_n \in \mathcal{R}_0(\Gamma_2)$ *with* $Y = \bigcup_1^n S_k$, *each* $\alpha|_{S_k}$ *has an affine extension, and* $\alpha(S_1), \ldots, \alpha(S_n)$ *are pairwise disjoint. Then with* $\tilde{\nu} : M(G_1) \to M(G_2)$ *the algebra homomorphism determined by* α, $\operatorname{rng} \tilde{\nu} = \tilde{\kappa}(\alpha)$ *if and only if* $\alpha(S_1), \ldots, \alpha(S_n)$ *are uniformly separated.*

Proof. Suppose $\operatorname{rng} \tilde{\nu} = \tilde{\kappa}(\alpha)$, let $1 \leq k \leq n$. Then by Theorem 1.3.2, there is a measure $\zeta_k \in M(G_2)$ such that $\hat{\zeta}_k = \chi_{S_k}$. It is now clear that $\zeta_k \in \tilde{\kappa}(\alpha)$, so that by hypothesis, there exists $\mu_k \in M(G_1)$ with $\tilde{\nu}(\mu_k) = \zeta_k$. Then $\widehat{\mu}_k$ is 1 on $\alpha(S_k)$ and 0 on $\alpha(S_j)$. Hence, by [5, Theorem 0.1], $\alpha(S_k)$ and $\alpha(S_j)$ are uniformly separated.

Conversely, if $\alpha(S_1), \ldots, \alpha(S_n)$ are uniformly separated, then by [5, Theorem 0.1], there exist $\xi_1, \ldots, \xi_n \in M(G_1)$ such that for each k, $\widehat{\xi}_k(\alpha(S_k)) = \{1\}$ and $\widehat{\xi}_k(\bigcup_{j \neq k} \alpha(S_j)) = \{0\}$.

Take $\mu \in \tilde{\kappa}(\alpha)$. Then by the arguments above, we have for each k that there exists $\mu_k \in M(G_1)$ such that $\hat{\mu} \circ (\alpha|_{S_k})^{-1} = \mu_k|_{\alpha(S_k)}$. Now put $\mu' = \sum_1^n \xi_k * \mu_k$. Then each $\gamma \in Y$ lies in some S_j, so

$$\widehat{\mu'} \circ \alpha(\gamma) = \sum_{k=1}^{n} \widehat{\xi}_k(\alpha(\gamma)) \cdot \hat{\mu}_k(\alpha(\gamma)) = \widehat{\mu_j}(\alpha_j(\gamma)) = \hat{\mu}(\gamma),$$

as required. ■

For the case where the $\alpha(S_1), \ldots, \alpha(S_n)$ are not disjoint, we start by showing that to determine whether $\operatorname{rng} \tilde{\nu} = \tilde{\kappa}(\alpha)$, it suffices to know $\alpha(Y)$, the range of α. This reformulation of the problem is based on Lemma 1.2.4, where we introduced the concept of looking for an extension for $\hat{f} \circ \alpha^{-1}$ $(f \in \kappa_{\mathcal{B}}(\alpha))$. In the case of homomorphisms between measure algebras, we are not considering the carrier spaces, as we were in Lemma 1.2.4, but the same idea applies.

Recall that by the construction in Example 1.6.6, a set $X \in \mathcal{R}_c(\Gamma)$ is a piecewise affine set, and so we have definitions of $A(X)$ and $B(X)$ from definition 1.6.4. The proof of Theorem 1.5.6 can then be divided into two stages—firstly showing that if $f \in \kappa(\alpha)$, then $\hat{f} \circ \alpha^{-1} \in A(\alpha(Y))$, and then showing that $A(\alpha(Y)) = A(\Gamma_1)|_{\alpha(Y)}$. A similar technique is applicable in the case where we have a homomorphism $\tilde{\nu} : M(G_1) \to M(G_2)$ that is obtained by extending $\nu : L^1(G_1) \to L^1(G_2)$.

A.2.3. Definition. Let $X \in \mathcal{R}_c(\Gamma)$. We say that X has the *Fourier-Stieltjes extension property (FSEP)* if $B(X) = B(\Gamma)|_X$.

A.2.4. Theorem. *Suppose* $\alpha : Y \to \Gamma_1$ *is a proper piecewise affine map. Then* $\operatorname{rng} \tilde{\nu} = \tilde{\kappa}(\alpha)$ *if and only if* $\alpha(Y) \in \mathcal{R}_c(\Gamma_1)$ *has FSEP.*

Proof. Throughout this proof, we will suppose $Y = S_1 \cup \cdots \cup S_n$, where each $S_k \in \mathcal{R}_0(\Gamma_2)$ is such that $\alpha|_{S_k}$ has $E_k = E_0(S_k)$ and a continuous affine extension $\alpha_k : E_k \to \Gamma_1$. We also take $\xi_k \in M(\Gamma_2)$ with $\hat{\xi}_k = \chi_{S_k}$.

Suppose $\alpha(Y)$ has FSEP and $\mu \in \tilde{\kappa}(\alpha)$, then $\mu * \xi_k \in \tilde{\kappa}(\alpha|_{S_k})$, for each $1 \leq k \leq n$. It follows from the discussion in the introduction to this section that $(\mu * \xi_k)\widehat{\ } \circ (\alpha|_{S_k})^{-1} \in B(\alpha(S_k))$. But $(\mu * \xi_k)\widehat{\ } \circ (\alpha|_{S_k})^{-1} = \hat{\mu} \circ \alpha^{-1}|_{\alpha(S_k)}$, and since $\alpha(Y) = \bigcup_1^n \alpha(S_k)$, we have $\hat{\mu} \circ \alpha^{-1} \in B(\alpha(Y))$. Thus by FSEP, there exists $\mu' \in M(G_1)$ with $\widehat{\mu'}|_{\alpha(Y)} = \hat{\mu} \circ \alpha^{-1}$, and then $\nu(\mu') = \mu$.

Conversely, if $\operatorname{rng} \hat{\nu} = \hat{\kappa}(\alpha)$, then any $F \in B(\alpha(Y))$ has $F \circ \alpha \in B(\Gamma_1)$, and clearly $F \circ \alpha^{-1} \in \hat{\kappa}(\alpha)$, so that $F \circ \alpha \in \operatorname{rng} \tilde{\nu}$. Let $\mu_0 \in M(G_1)$ be such that $\tilde{\nu}(\mu_0)\widehat{\ } = F \circ \alpha$, then $\hat{\mu}_0 \circ \alpha = F \circ \alpha$, so that $\hat{\mu}_0|_{\alpha(Y)} = F$. ∎

We can now generalize the argument in Theorem A.2.2 to give the following result.

A.2.5. Theorem. *Suppose $X \in \mathcal{R}_c(\Gamma)$ is a piecewise affine set, say $X = \bigcup_1^n X_k$, for some disjoint $X_1, \ldots, X_n \in \mathcal{R}_c(\Gamma)$ such that each X_k is in the coset ring of a closed coset E_k. Then X has FSEP if and only if $X_1, \ldots, X_n$ are uniformly separated.* ∎

This theorem now allows us to characterize FSEP for sets $X \in \mathcal{R}_c(\Gamma)$ that are discrete. This leads to a result (Corollary A.2.9) suggesting a link between the property of $X \in \mathcal{R}_c(\Gamma)$ having FSEP and the property of $\mathcal{I}(X)$ having a Banach space complement, as investigated in the papers [1, 2, 3, 4]. This link will be further investigated later in this appendix.

A.2.6. Definition. A set $X \subseteq \Gamma$ is *uniformly discrete* if there is a neighbourhood of $e \in \Gamma$ such that for any $x \in X$, $(x + U) \cap X = \{x\}$.

A.2.7. Lemma. *If $X \subseteq \Gamma$, the following are equivalent :*

(i) X is uniformly discrete,

(ii) *e is an isolated point of $X - X$, and*

(iii) *if U and V are disjoint subsets of X, then U and V are uniformly separated.*

Proof. *Clear.* ∎

A.2.8. Theorem. *A discrete set $X \in \mathcal{R}_c(\Gamma)$ has FSEP if and only if X is uniformly discrete.*

Proof. By [4, Lemma 2.2], for any discrete $X \in \mathcal{R}_c(\Gamma)$, there exist discrete cosets $E_1, \ldots, E_n$ in Γ and for each k, some $S_k \in \mathcal{R}(E_k)$, such that $X = \bigcup_1^n S_k$. By Theorem A.2.5, it suffices to show that X is uniformly discrete if and only if $S_1, \ldots, S_n$ are uniformly separated. The "only if" part of this follows from the implication (i) $\Longrightarrow$ (iii) in Lemma A.2.7. Supposing $S_1, \ldots, S_n$ are uniformly separated, then there exists $U \subseteq \Gamma$, a neighbourhood of e, such that for each j, k, with $j \neq k$, $(S_j + U) \cap S_k = \varnothing$. Then

$$U \cap (X - X) = U \cap \Big(\bigcup_1^n (S_k - S_k)\Big) \subseteq U \cap \Big(\bigcup_1^n (E_k - E_k)\Big),$$

and each $E_k - E_k$ is a discrete subgroup of Γ, so that e is an isolated point of $X - X$. Hence, by Lemma A.2.7, X is uniformly discrete. ∎

A.2.9. Corollary. *A discrete set $X \in \mathcal{R}_c(\Gamma)$ has FSEP if and only if $\mathcal{I}(X)$ has a Banach space complement in $L^1(G)$.*

Proof. Suppose $X = S_1 \cup \cdots \cup S_n$, where for each k, $S_k \in \mathcal{R}(E_k)$ and E_k is a discrete coset. By [4, Theorem 2.3], $\mathcal{I}(X)$ is complemented if and only if the S_k are uniformly separated. We have seen that this occurs if and only if X has FSEP. ∎

A.3. The Fourier-Stieltjes Extension Property in a Pair of Subgroups

In this section we characterize the Fourier-Stieltjes Extension Property in sets $X \in \mathcal{R}_c(\Gamma)$ of the form $X = \Lambda \cup \Xi$, where Λ and Ξ are closed subgroups of Γ. By translation, this also characterizes FSEP for sets that are the union of a pair of intersecting closed cosets. Since we have already characterized FSEP for a pair of non-intersecting cosets in Theorem A.2.5, we can completely describe FSEP for sets that are the union of a pair of closed cosets. As one may expect, uniform separation again makes an appearance.

We start with a general result on FSEP.

A.3.1. Lemma. *Suppose $X_1, X_2 \in \mathcal{R}_c(\Gamma)$ and X_2 has FSEP. Then $X_1 \cup X_2$ has FSEP if and only if any $F \in \mathcal{I}_{B(X_1)}(X_1 \cap X_2)$ has an extension in $\mathcal{I}_{B(\Gamma)}(X_2)$.*

Proof. Suppose $\mathcal{I}_{B(X_1)}(X_1 \cap X_2) = \mathcal{I}_{B(\Gamma)}(X_2)|_{X_1}$ and $F \in B(X_1 \cup X_2)$. Since X_2 has FSEP, and $F|_{X_2} \in B(X_2)$ there exists $F_1 \in B(\Gamma)$ such that $F_1|_{X_2} = F|_{X_2}$. Then $F|_{X_1} - F_1|_{X_1} \in \mathcal{I}_{B(X_1)}(X_1 \cap X_2)$, so by hypothesis, there exists $F_2 \in \mathcal{I}_{B(\Gamma)}(X_2)$ with $F|_{X_1} - F_1|_{X_1} = F_2|_{X_1}$. Then $F_1 + F_2 \in B(\Gamma)$, $(F_1 + F_2)|_{X_1} = F|_{X_1}$ and $(F_1 + F_2)|_{X_2} = F|_{X_2}$. Hence $X_1 \cup X_2$ has FSEP.

Conversely, if $X_1 \cup X_2$ has FSEP and $F \in \mathcal{I}_{B(X_1)}(X_1 \cap X_2)$, then we can extend F to $F_1 \in B(X_1 \cup X_2)$ by setting $F_1(X_2) = \{0\}$. Then by FSEP, F_1 extends to $F_2 \in B(\Gamma)$, which is the desired extension of F. ∎

Thus, if we are considering whether the union of a pair of subgroups $\Lambda \cup \Xi$ has FSEP, we need to be able to extend any $F \in \mathcal{I}_{B(\Lambda)}(\Lambda \cap \Xi)$ to $F' \in \mathcal{I}_{B(\Gamma)}(\Xi)$. For this, we need to examine the standard method of extending a Fourier-Stieltjes transform on a closed subgroup of a locally compact Abelian group to a Fourier-Stieltjes transform on the entire group. We take a definition from [32, 8.1.9], where the non-Abelian case is also considered.

A.3.2. Definition. Suppose G is a locally compact Abelian group with a closed subgroup H. We call a function $\beta \in C_b^+(G)$ a *Bruhat function for the quotient* $G \to G/H$ if for any compact $C \subseteq G$, we have $\beta|_{C+H} \in C_{00}(C+H)$, and for each $x \in G$, $\int_H \beta(x+\xi)\, d\xi = 1$. (The first condition here is equivalent to $\beta|_{C+H} \in C_{00}(G)|_{C+H}$, the criterion given in [32, 8.1.9], by a simple application of the Tietze Extension Theorem.) The existence of such β for any closed subgroup of a locally compact Abelian group is shown in [32, 8.1.9]. For such β, we have a continuous linear mapping $T_{H,\beta} : C_0(G) \to C_0(G/H)$ given by

$$T_{H,\beta}(\psi)(x+H) = \int_H \psi(x+y)\beta(x+y)\, d\lambda_H(y).$$

Moreover, by [32, 8.2.7], the above formula defines a continuous linear mapping $C_b(G) \to C_b(G/H)$. This extension will also be denoted $T_{H,\beta}$.

Let $T^*_{H,\beta} : M(G/H) \to M(G)$ be the adjoint of $T_{H,\beta}$. As stated in [32, 8.2.7], $T^*_{H,\beta}$ is a right inverse for $\tilde{T}_H : M(G) \to M(G/H)$, and since $\tilde{T}_H$ is determined by $\left(\tilde{T}_H(\mu)\right)^{\widehat{}} = \hat{\mu}|_\Lambda$, the function $T^*_{H,\beta}$ extends Fourier-Stieltjes transforms. Moreover, it is shown in [32, 8.2.7], that if $\psi \in C_b(G/H)$ and $\mu \in M(G/H)$ then $\int_G \psi\, d(T^*_{H,\beta}) = \int_{G/H} T_{H,\beta}(\psi)\, d\mu$. We will show that if we can find a Bruhat function β for the quotient $G \to G/H$ that is constant on cosets of C, then $T^*_{H,\beta}$ maps $\mathcal{I}_{B(\Lambda)}(\Lambda \cap \Xi)$ into $\mathcal{I}_{B(\Gamma)}(\Xi)$.

We first establish some results on using $\tilde{T}_H\mu$ to integrate a function in $C_b(G/H)$.

A.3.3. Lemma. *Suppose G is a locally compact Abelian group with closed subgroup H. If $\psi \in C_b(G/H)$ and $\mu \in M(G)$, then*

$$\int_G \psi\, d(\tilde{T}_H\mu) = \int_G \psi \circ Q_H\, d\mu.$$

Proof. For $\mu \in M^+(G)$ and $\psi \in C_b^+(G/H)$, we have by the regularity of μ and

$\tilde{T}_H\mu$, that

$$\int_{G/H} \psi \, d(\tilde{T}_H\mu) = \sup\left\{\int_{G/H} \psi_0 \, d(\tilde{T}_H\mu) : \psi_0 \in C_0^+(G/H),\ \psi_0 \le \psi\right\}$$

$$= \sup\left\{\int_G \psi_1 \, d\mu : \psi_0 \in C_0^+(G/H),\ \psi_1 \in C_0^+(G),\ \psi_1 \le \psi_0 \circ Q_H \le \psi \circ Q_H\right\}$$

and $\displaystyle\int_G \psi \circ Q_H \, d\mu = \sup\left\{\int_G \psi_1 \, d\mu : \psi_1 \in C_0^+(G), \psi_1 \le \psi \circ Q_H\right\}$.

But if $\psi_1 \in C_0^+(G)$ has $\psi_1 \le \psi \circ Q_H$, then $\psi_2(x+H) = \max_{y\in H} \psi_1(x+y)$ defines $\psi_2 \in C_0^+(G/H)$ with $\psi_1 \le \psi_0 \circ Q_H \le \psi \circ Q_H$. Thus the above two integrals are equal. The general case follows by the Hahn-Jordan Decomposition Theorem. ■

A.3.4. Corollary. *Suppose* $\mu \in M(G)$ *has* $\hat{\mu}|_\Lambda = 0$. *If* $\psi \in C_b(G)$ *is constant on cosets of* H, *then* $\int_G \psi \, d\mu = 0$. ■

A.3.5. Definition. [4, 3.6] Suppose Λ, Ξ are closed subgroups of Γ. We say (Λ, Ξ) satisfies **(D)** if $(\Lambda + \Xi)/(\Lambda \cap \Xi) = \Lambda/(\Lambda \cap \Xi) \oplus \Xi/(\Lambda \cap \Xi)$. (Remember that here, "$\oplus$" means that we have a *topological* direct sum.)

A.3.6. Theorem. *Suppose* Λ *and* Ξ *are closed subgroups of a locally compact Abelian group* G, *with annihilators* H *and* K *respectively. Then the following are equivalent :*

(i) (Λ, Ξ) *satisfies* **(D)**,

(ii) (H, K) *satisfies* **(D)**,

(iii) $\Lambda \cup \Xi$ *has FSEP, and*

(iv) $\mathcal{I}(\Lambda \cup \Xi)$ *has a Banach space complement in* $L^1(G)$.

Proof. The equivalences (i) $\iff$ (ii) $\iff$ (iv) are proven in [4].

Suppose $\Lambda \cup \Xi$ has FSEP. We show that $\pi : \Lambda/(\Lambda \cap \Xi) \times \Xi/(\Lambda \cap \Xi) \to (\Lambda + \Xi)/(\Lambda \cap \Xi)$, given by $\pi\big(\lambda + (\Lambda \cap \Xi), \xi + (\Lambda \cap \Xi)\big) = \lambda + \xi + (\Lambda \cap \Xi)$ is bicontinuous. It is clearly continuous.

Let $U_1 \subseteq \Lambda/(\Lambda \cap \Xi)$ be a neighbourhood of e, and let $f_0 \in A(\Lambda/(\Lambda \cap \Xi))$ have support in U_1 and $f_0(e) = 1$. Define $f \in B(\Lambda \cup \Xi)$ by $f|_\Lambda = f_0 \circ Q_{\Lambda\cap\Xi}$ and $f|_\Xi = 1$. By FSEP, there exists $\mu \in M(G)$ with $\hat{\mu}|_{\Lambda\cup\Xi} = f$. Then with $U_1' = Q^{-1}_{\Lambda\cap\Xi}(U_1)$, we have $\hat{\mu}(\Xi) = \{1\}$ and $\hat{\mu}(\Lambda \setminus U_1') = \{0\}$. Hence Ξ and $\Lambda \setminus U_1'$ are uniformly separated. Let $V_1 \subseteq \Gamma$ be a neighbourhood of e such that $(\Xi + V) \cap (\Lambda \setminus U_1') = \varnothing$. Then $(V_1 + \Xi) \cap \Lambda \subseteq U_1'$ so $V_1 \cap (\Lambda + \Xi) \subseteq U_1' + \Xi$.

Similarly, if $U_2 \subseteq \Xi/(\Lambda \cap \Xi)$ is a neighbourhood of e, there exists $V_2 \subseteq \Gamma$, a neighbourhood of e with $V_2 \cap (\Lambda + \Xi) \subseteq U_2' + \Lambda$. Put $V = V_1 \cap V_2$, then V is a neighbourhood of e with $V \cap (\Lambda + \Xi) \subseteq (U_1' + \Xi) \cap (U_2' + \Lambda)$. Moreover, $U_1' \subseteq \Lambda$ and $U_2' \subseteq \Xi$, so $(U_1' + \Xi) \cap (U_2' + \Lambda) \subseteq U_1' + U_2' + (\Lambda \cap \Xi)$. Hence $Q_{\Lambda\cap\Xi}\big(V \cap (\Lambda + \Xi)\big)$, a neighbourhood of e in $(\Lambda + \Xi)/(\Lambda \cap \Xi)$ is a subset of $\pi(U_1 + U_2)$, so that π is open.

Conversely, suppose (Λ, Ξ) satisfies **(D)**. Note that if we put $\Gamma' = \overline{\Lambda + \Xi} \subseteq \Gamma$, then $B(\Gamma') = B(\Gamma)|_{\Gamma'}$, so it is clearly sufficient to prove that $\Lambda \cup \Xi$ has FSEP in the case $\overline{\Lambda + \Xi} = \Gamma$. In this case, $H \cap K = \{e\}$, and since (H, K) satisfies **(D)**, we have $H + K = H \oplus K$.

Let $\beta' \in C_b(G/K)$ be a Bruhat function for the quotient

$$G/K \to (G/K) \Big/ ((H + K)/K) \cong G/(H + K),$$

and let $\beta = \beta' \circ Q_K \in C_b(G)$. Then $H' = (H + K)/K \cong H$, so providing we choose Haar measures on all subgroups appropriately, we have for each $s \in K$ that $\int_H \beta(x + \xi)d\lambda_H(\xi) = \int_{H'} \beta'\big((x + K) + \eta\big)d\lambda_{H'}(\eta) = 1$. Also, if $C \subseteq G$ is compact, then $\{\, x \in C + H : \beta(x) \neq 0 \,\} \subseteq (C + H) \cap Q_K^{-1}\{\, x + K \in Q_K(C) : \beta'(x + K) \neq 0 \,\}$, which is compact, since $H + K = H \oplus K$. Hence $\beta \in C_b(G)$ is a Bruhat function for the quotient $G \to G/H$, and β is constant on cosets of K.

Thus if $\mu_0 \in M(G/H)$ has $\hat{\mu}_0(\Xi \cap \Lambda) = \{0\}$, then for each $\xi \in \Xi$, $T_{H,\beta}(\bar{\xi})$ is constant on cosets of K/H, so that by Corollary A.3.4, $\int_{G/H} T_{H,\beta}(\xi)\, d\mu_0 = 0$. Hence $\mu = T^*_{H,\beta}(\mu_0)$ has $\hat{\mu}|_\Lambda = \mu_0$ and $\hat{\mu}|_\Xi = 0$. This is sufficient for FSEP, by Lemma A.3.1. ■

A.4. Factorization in Ideals of Measure Algebras

From Corollary A.2.9 and Theorem A.3.6, it seems that $X \in \mathcal{R}_c(\Gamma)$ has FSEP if and only if $\mathcal{I}(X)$ is complemented. However, this is not the case, as we will see later in this section. As the title of this section suggests, this construction relies on factorization in ideals of measure algebras.

A.4.1. Lemma. *Suppose $X_1, X_2, X_3 \in \mathcal{R}_c(\Gamma)$, are such that each of $X_1 \cup X_2$, $X_1 \cup X_3$ and $X_2 \cup X_3$ has FSEP. If $\mathcal{I} = \mathcal{I}_{B(X_1)}(X_1 \cap (X_2 \cup X_3))$ factors, then $X_1 \cup X_2 \cup X_3$ has FSEP.*

Proof. By Lemma A.3.1, it is sufficient to extend $F \in \mathcal{I}$ to $F' \in \mathcal{I}_{B(\Gamma)}(X_2 \cup X_3)$. However, if we can factor $F = F_1F_2$, where $F_1, F_2 \in \mathcal{I}$, then $F_1 \in \mathcal{I}_{B(X_1)}(X_1 \cap X_2)$ and $F_2 \in \mathcal{I}_{B(X_1)}(X_1 \cap X_3)$, and so by Lemma A.3.1, we can extend F_1 and F_2 to $F_1' \in \mathcal{I}_{B(\Gamma)}(X_2)$ and $F_2' \in \mathcal{I}_{B(\Gamma)}(X_3)$ respectively. Then $F' = F_1'F_2' \in \mathcal{I}_{B(\Gamma)}(X_2 \cup X_3)$ is an extension of F, as required. ∎

A.4.2. Corollary. *Suppose $X_1, X_2, X_3 \in \mathcal{R}_c(\Gamma)$, are such that each of $X_1 \cup X_2$, $X_1 \cup X_3$ and $X_2 \cup X_3$ has FSEP. If $\mathcal{I}_{B(X_1)}(X_1 \cap X_2)$ and $\mathcal{I}_{B(X_1)}(X_1 \cap X_2)$ each has bounded approximate identity, then $X_1 \cup X_2 \cup X_3$ has FSEP.*

Proof. If $\mathcal{I}, \mathcal{J}$ are ideals of a commutative Banach algebra $\mathfrak{A}$, each with bounded approximate identity, then the term-by-term product of the bounded approximate identities, with the product directed set, is a bounded approximate identity for $\mathcal{I} \cap \mathcal{J}$. ∎

We can now apply the above to the set $X = (\mathbb{R} \times \mathbb{Z}) \cup (\mathbb{Z} \times \mathbb{R}) \cup {}_\theta\mathbb{R} \subseteq \mathbb{R}^2$, where ${}_\theta\mathbb{R} = \{ (x, y) \in \mathbb{R}^2 : x \sin\theta = y \cos\theta \}$. It was shown in [4, Example 0.1(iii)], that for such X, $\mathcal{I}(X)$ is complemented if and only if $\tan\theta$ is rational. However, such X always has FSEP.

A.4.3. Proposition. *If* $X = (\mathbb{R} \times \mathbb{Z}) \cup (\mathbb{Z} \times \mathbb{R}) \cup {}_\theta\mathbb{R} \subseteq \mathbb{R}^2$, *then* X *has FSEP.*

Proof. Put $\Lambda_1 = \mathbb{R} \times \mathbb{Z}$, $\Lambda_2 = \mathbb{Z} \times \mathbb{R}$ and $\Lambda_3 = {}_\theta\mathbb{R} \subseteq \mathbb{R}^2$, then each of the pairs (Λ_1, Λ_2), (Λ_1, Λ_3), and (Λ_2, Λ_3) satisfy **(D)**. Moreover, the ideals $\mathcal{I}_{B({}_\theta\mathbb{R})}(\Lambda_1 \cap \Lambda_3)$ and $\mathcal{I}_{B({}_\theta\mathbb{R})}(\Lambda_2 \cap \Lambda_3)$ are both isomorphic to $\mathcal{I}_{M(\mathbb{R})}(\mathbb{Z})$. But $\mathcal{I}_{M(\mathbb{R})}(\mathbb{Z})$ has bounded approximate identity, $\{e_n\}_{n\in\mathbb{N}}$, where $e_n = \delta_e - \frac{1}{2n+1}\sum_{-n}^{n} \delta_k$. Hence, by Corollary A.4.2, X has FSEP. ■

The method of factoring ideals of measure algebras in the above case generalizes, so that we can prove that for $E_1, \ldots, E_n$ closed cosets such that any pair (E_i, E_j) is either uniformly separated, or satisfies (D). This, however, does not cover [4, Example 0.1(v)] or [1, Example 4.1], which we can prove to have FSEP by following the same order of construction that was used to prove the ideal in the group algebra to be complemented.

I finish with some conjectures.

A.4.4. Conjecture. *If* $X \in \mathcal{R}_c(\Gamma)$ *and* $\mathcal{I}(X)$ *is complemented, then* X *has FSEP.*

A.4.5. Conjecture. *If* $X, Y \in \mathcal{R}_c(\Gamma)$ *are such that* $\mathcal{I}(X)$, $\mathcal{I}(Y)$, *and* $\mathcal{I}(X \cap Y)$ *are complemented, and* $X \cup Y$ *has FSEP, then* $\mathcal{I}(X \cup Y)$ *is complemented.*

Even being able to prove this in the case where Y is a closed subgroup (or coset) would be useful. The similarities in the proofs of Proposition A.3.6 and [4, Theorem 4.4] are encouraging.

A.5. The Gel'fand Transform

It should be noted that there is another natural way in which we might be able to characterize the range of a homomorphism between measure algebras, which is possibly more natural than the subalgebra $\tilde{\kappa}(\alpha)$. A conclusive result may hold if we instead consider the Gel'fand transform and apply Lemma 1.2.1 to the homomorphism $\tilde{\nu} : M(G_1) \to M(G_2)$. This gives us $\tilde{Y} = \Phi_{M(G_2)} \setminus Z_{M(G_2)}(\operatorname{rng} \tilde{\nu})$ and $\tilde{\alpha} = \tilde{\nu}^*|_{\tilde{Y}} : \tilde{Y} \to \Phi_{M(G_1)}$. Then for α as above, $\kappa_{M(G_2)}(\tilde{\alpha}) \subseteq \tilde{\kappa}(\alpha)$. Hence there is a possibility that $\operatorname{rng} \tilde{\nu} = \kappa_{M(G_2)}(\tilde{\alpha})$ will always hold. Considerations of this sort are difficult, as they involve the carrier spaces of measure algebras. These are quite inaccessible.

Appendix B. Banach Space Complements of Ideals in Group Algebras

Let G be a locally compact Abelian group, with the group operation represented additively, and let $\mathcal{I}$ be a closed ideal in $L^1(G)$. The problem of the existence of a Banach space complement to an $\mathcal{I}$ in $L^1(G)$, is equivalent to that of finding a continuous linear projection $L^1(G) \to \mathcal{I}$. (We are not looking for a multiplicative projection—such ideals we have already classified in Theorem 1.7.2.) Of the investigations into the existence of such a projection, the methods in the paper [4] of D. Alspach, A. Matheson and J. Rosenblatt, provide the most exhaustive methods of constructing such a projection known to the author. The paper [4] concludes with a statement that the authors knew of no example of a complemented ideal for which the procedure could not be applied. The question as to the existence of such an ideal was posed explicitly in [2, Question 4.1], at least in a weaker form. Here it was asked whether there was an ideal for which the "natural" ways of applying the methods of [4] fail.

In this appendix, we construct two examples of complemented ideals of group algebras. The first answers the question [2, 4.1] affirmatively, yet it does yield to the methods of [4] in a less-than-natural manner. The second ideal constructed does not yield to the methods in [4].

B.1. Background

The starting point for the consideration of the problem of determining whether $\mathcal{I}$ is complemented is the paper [36] of H.P. Rosenthal, where it was shown that if $\mathcal{I}$ is complemented, then $Z(\mathcal{I}) \in \mathcal{R}_d(\Gamma)$. Since a hull is closed, we have $Z(\mathcal{I}) \in \mathcal{R}_c(\Gamma)$. (An alternative approach is given in [28, Theorem 2], where it is shown that a complemented ideal in a commutative group algebra has bounded approximate identity.) Since such a hull is a set of synthesis, the complemented ideals of $L^1(G)$ can be classified in terms of their hulls. The problem is now to find suitable algebraic, topological and combinatoric conditions on $X \in \mathcal{R}_c(\Gamma)$ to characterize when $\mathcal{I}(X)$ is complemented.

In [3], Alspach and Matheson characterized the complemented ideals in $L^1(\mathbb{R})$ in terms equivalent to the uniform separation criterion we saw in Appendix A. This case was particularly simple, as any proper closed coset in $\mathbb{R}$ is either a singleton or a translate of some $\xi\mathbb{Z}$, and so any $X \in \mathcal{R}_c(\mathbb{R})$ such that $X \neq \mathbb{R}$ is discrete. Moreover, uniform separation of such cosets is easily characterized in algebraic terms. A solution for general locally compact Abelian G is far from being a simple generalization of this result for $\mathbb{R}$. In [4], Alspach, Matheson and Rosenblatt developed a procedure that, given *certain* $X \in \mathcal{R}_c(\Gamma)$, constructed a continuous projection, thus showing $\mathcal{I}(X)$ to be complemented. It was further shown in [1] that for the case $G = \mathbb{R}^2$, this construction succeeds if and only if $\mathcal{I}(X)$ is complemented.

The exact criteria established in [4] will not be discussed here—they are quite complicated. The examples discussed in Appendix A are indicative of some of the simple ways in which a hull $X \in \mathcal{R}_c(\Gamma)$ can have complemented or uncomplemented kernel, in that there seems to be some involvement of uniform separation. Further instances of this will be seen in the examples in the next section, and in the two examples around which this appendix is based. The aspect of the construction in [4] that is crucial for our purposes is the manner in which the projection is constructed. This is done by taking a decomposition of $X \in \mathcal{R}_c(\Gamma)$ as was achieved by Schreiber in [38, Theorem 1.7], that is, $X = \bigcup_1^n X_k$, where each X_k is in the coset ring of some closed coset $E_k \subseteq \mathcal{R}(\Gamma)$. If such a decomposition can be found with $\mathcal{I}(X_1 \cup \cdots \cup X_k)$ complemented in $\mathcal{I}(X_1 \cup \cdots \cup X_{k-1})$ for each $1 \leq k \leq n$, then we can construct a projection $L^1(G) \to \mathcal{I}(X)$ simply by composing the chain of projections

$$L^1(G) = \mathcal{I}(\emptyset) \to \mathcal{I}(X_1) \to \cdots \to \mathcal{I}(X_1 \cup \cdots \cup X_{n-1}) \to \mathcal{I}(X).$$

Some features of this construction suggest that there may be difficulties with working with it in general. In particular, it imposes an order, which we call an *order of assembly* of $X_1, \ldots, X_n$. Moreover, for certain hulls, the order of assembly is critical in constructing the desired chain of projections. Examples of this are [4, Example 0.1(v)] and [1, Example 4.1], which appear below, slightly modified, as Examples B.2.3, and B.2.4.

The other point to note is that "uncomplementary" behaviour of a hull tends to be a localized phenomenon, in that if $X \in \mathcal{R}_c(\Gamma)$ is the hull of an uncomplemented ideal, then there is a set $X_0 \in \mathcal{R}_c(\Gamma)$ such that if U is a neighbourhood of X_0 and $X' \cap U = X \cap U$, then $\mathcal{I}(X')$ is uncomplemented. It tends to be around this set X_0 that the failure of some sort of uniform separation occurs. This observation will be formalized and used below.

In Sections B.3 and B.5 we use these two features to construct two hulls $X \in \mathcal{R}_c(\Gamma)$ with complemented kernel, but for which there is no order of assembly that gives a chain of projections. The first example is a union of closed subgroups $\Lambda_1, \ldots, \Lambda_n$ such that the behaviour of $X = \bigcup_1^n \Lambda_k$ near certain subgroups $\lambda_1, \ldots, \lambda_m$ is similar to that of certain basic examples for which the order of assembly is critical. More precisely, for $1 \leq k \leq m$, if we let $\mathbb{J}_k$ be those j for which Λ_j intersects λ_k (other than at 0), and let $X_k = \bigcup_{j \in \mathbb{J}_k} \Lambda_j$, then the construction is such that $\mathcal{I}(X_k)$ is an instance of one of the basic examples in Section B.2 which is complemented but for which certain orders of assembly do not give a chain of projections. Moreover, the example is set up so that any overall ordering of $\{1, \ldots, n\}$ will give an ordering on at least one of the sets $\mathbb{J}_k$ for which there is no chain of projections.

Section B.4 introduces some methods which enable the demonstration of complementedness of this $\mathcal{I}(X)$ and the non-existence of a chain of projections based on an ordering of the subgroups $\Lambda_1, \ldots, \Lambda_n$. This provides an explicit answer to the question [2, 4.1] of D.E. Alspach. These methods involve considering the behaviour of each part of a such a complicated set X in a local manner, and then using a partition of the identity to combine the separate parts.

The example of Section B.3 does not demonstrate conclusively the inadequacy of the methods of [4], as it is possible to construct a chain of projections as above, where each X_k is a proper subset of some Λ_j. Thus, if we partition each Λ_j into smaller pieces, each in $\mathcal{R}(\Lambda_j)$, and then assemble these pieces in a particular order, we can find a chain of projections. It is therefore desirable to find an example for which we cannot even accomplish this. Section B.5 is devoted to constructing such an example, and so completes the demonstration of the need for more complete methods than those of [4].

B.2. Basic Examples

We have the following examples, based on previously-known results.

B.2.1. Example. Let $E_1, \ldots, E_n$ be Euclidean cosets in $\mathbb{R}^n$ and put $X = \bigcup_1^n E_k$. Then $\mathcal{I}(X)$ is complemented. (By [4, Theorem 3.12.])

B.2.2. Example. For $\xi_1, \xi_2 \in \mathbb{R}^+$, and linearly independent $a, b, c \in \mathbb{R}^3$, let $\Lambda_1 = \xi_1\mathbb{Z}a + \mathbb{R}b$ and $\Lambda_2 = \xi_2\mathbb{Z}a + \mathbb{R}c$. Then $\mathcal{I}(\Lambda_1 \cup \Lambda_2)$ is complemented if and only if ξ_1 and ξ_2 are rationally dependent. (By [4, Theorems 3.11 and 4.4.])

B.2.3. Example. As in Example B.2.2, put $\Lambda_3 = \mathbb{R}a$, then $\mathcal{I}(\Lambda_1 \cup \Lambda_2 \cup \Lambda_3)$ is complemented. (A trivial generalization of [4, Example 0.1 (v)].) There exist projections $\mathcal{I}(\Lambda_1) \to \mathcal{I}(\Lambda_1 \cup \Lambda_3)$, $\mathcal{I}(\Lambda_2) \to \mathcal{I}(\Lambda_2 \cup \Lambda_3)$, $\mathcal{I}(\Lambda_3) \to \mathcal{I}(\Lambda_1 \cup \Lambda_3)$, and $\mathcal{I}(\Lambda_3) \to \mathcal{I}(\Lambda_2 \cup \Lambda_3)$, but not $\mathcal{I}(\Lambda_1) \to \mathcal{I}(\Lambda_1 \cup \Lambda_2)$, or $\mathcal{I}(\Lambda_2) \to \mathcal{I}(\Lambda_1 \cup \Lambda_2)$, unless $\xi_1/\xi_2 \in \mathbb{Q}$. Thus, if $\xi_1/\xi_2 \notin \mathbb{Q}$, any order of assembly must add Λ_1 or Λ_2 last.

B.2.4. Example. As in Example B.2.2, let $d, e \in \mathbb{R}^3$ be such that $d \notin \operatorname{span}\{a, b\} \cup \operatorname{span}\{a, c\}$ and $e \in \operatorname{span}\{a, d\} \setminus (\mathbb{R}a \cup \mathbb{R}d)$. Put $\Lambda_4 = \xi_1\mathbb{Z}a + \mathbb{R}d$ and $\Lambda_5 = \xi_2\mathbb{Z}a + \mathbb{R}e$, then $\mathcal{I}(\Lambda_1 \cup \Lambda_2 \cup \Lambda_4 \cup \Lambda_5)$ is complemented. (Here we have three planes, $\operatorname{span}\{a, b\}$, $\operatorname{span}\{a, c\}$, and $\operatorname{span}\{a, d\} = \operatorname{span}\{a, e\}$, with a common intersection $\mathbb{R}a$, such that each of the first two planes contains a set of parallel lines, whilst the third plane contains a parallelogram grid with sides parallel to $\mathbb{R}d$ and $\mathbb{R}e$.) This is a trivial reworking of [1, Example 4.1], and the proof of complementedness remains the same.

Again, in the case $\xi_1/\xi_2 \notin \mathbb{Q}$, the only orders of assembly that lead to a chain of projections are those which add Λ_1 or Λ_2 last.

These last three examples each retains the same properties if $\mathbb{Z}$ is replaced by $\widetilde{\mathbb{Z}} = \mathbb{Z} + 1/2$ in the definitions of Λ_1, Λ_2, Λ_4, and Λ_5. This fact will be used in the example in Section B.5.

B.3. Building a Hull in $\mathbb{R}^3$

Consider a Euclidean group $\mathbb{R}^n$. We will often refer to elements of $\mathbb{R}^n$ as *points*. We will also use the terms *line*, *grille*, and *plane* to refer to closed cosets in $\mathbb{R}^n$ that are affinely homeomorphic to $\mathbb{R}$, $\mathbb{R}\times\mathbb{Z}$, and $\mathbb{R}^2$, respectively. These will generally be of the form $\mathbb{R}x+z$, $\mathbb{Z}x+\mathbb{R}y+z$ and $\mathbb{R}x+\mathbb{R}y+z$, respectively, where $x, y, z \in \mathbb{R}^n$ and $\{x, y\}$ is linearly independent. (In particular $x \neq 0$ and $y \neq 0$.) If x and $y \neq 0$ are two linearly dependent vectors, we will use x/y for the value $\xi \in \mathbb{R}$ such that $\xi y = x$. If $x, y \in \mathbb{R}^n$ are linearly independent, and $z \in (\mathbb{R}x+\mathbb{R}y) \setminus \mathbb{R}y$, say $z = \xi x + \zeta y$, then for any nonzero $\eta \in \mathbb{R}$, $(\eta\mathbb{Z}x+\mathbb{R}y)\cap\mathbb{R}z = |\eta/\xi|\mathbb{Z}z$. Consequently, if x, y, and z are colinear, then $\xi+\zeta=1$ and $\xi = (z-y)/(x-y)$. Hence

$$(\eta\mathbb{Z}x+\mathbb{R}y)\cap\mathbb{R}z = \eta\left|\frac{x-y}{z-y}\right|\mathbb{Z}z$$

We will often define a line in terms of points lying in it, for instance, if $a, b, c, d, \ldots$ are distinct points all lying in a line λ, then $\overline{ab}$, $\overline{bd}$, $\overline{abc}$, $\overline{abcd}$, ... are all possible descriptions of λ.

Define points $P = \{a, \ldots, i\} \subseteq \mathbb{R}^2 \times \{1\} \subseteq \mathbb{R}^3$ as follows :

$$\begin{aligned}
a &= (\ \ 0,\ 42,1), & d &= (\ \ 0,\ 21,1), & g &= (\ 14,\ 14,1),\\
b &= (\ 42,\ \ 0,1), & e &= (\ 21,\ \ 0,1), & h &= (\ \ 0,-14,1),\\
c &= (-42,-42,1), & f &= (-21,-21,1), & i &= (-14,\ \ 0,1),
\end{aligned}$$

we then have lines $\lambda_a = \overline{age}$, $\lambda_b = \overline{bhf}$, $\lambda_c = \overline{cid}$, $\lambda_d = \overline{dgb}$, $\lambda_e = \overline{ehc}$, $\lambda_f = \overline{fia}$, $\lambda_g = \overline{gfc}$, $\lambda_h = \overline{hda}$, and $\lambda_i = \overline{ieb}$. Note that if $x \in P$, then $x \in \lambda_x$ and $\{p \in P : x \in \lambda_p\}$ has three elements, as does $\{p \in P : p \in \lambda_x\} = \lambda_x \cap P$. (This is a solution to the tree-planting problem "plant 9 trees so that there are nine rows of three trees in each, each tree being in three rows." Such problems often occur in recreational mathematics.) A sketch of this figure would probably be quite useful to the reader. For each $p \in P$, let $\pi_p = \operatorname{span} \lambda_p$, the plane containing 0 and λ_p. The lines $\{\lambda_p : p \in P\}$ have seven extra points $\{t, \ldots, z\} = P'$ at which there are pairwise intersections. These are

$$\begin{aligned}
t &= (\ \ \ 0, \ \ \ 0,1) = \overline{gfc} \cap \overline{hda} \cap \overline{ieb}, \\
u &= (\ \ \ 6, \ 30,1) = \overline{age} \cap \overline{cid}, \\
v &= (\ 24, -6,1) = \overline{age} \cap \overline{bhf}, \\
w &= (-30,-24,1) = \overline{bhf} \cap \overline{cid}, \\
x &= (\ 30, \ \ \ 6,1) = \overline{dgb} \cap \overline{ehc}, \\
y &= (-24,-30,1) = \overline{ehc} \cap \overline{fia}, \\
z &= (\ -6, \ 24,1) = \overline{fia} \cap \overline{dgb}.
\end{aligned}$$

Now take $\xi_1, \xi_2, \xi_3 \in \mathbb{R}$ pairwise rationally independent, and put

$$\begin{aligned}
\Lambda_a &= \xi_1 \mathbb{Z} u + \mathbb{R} a, & \Lambda_d &= \xi_2 \mathbb{Z} x + \mathbb{R} d, & \Lambda_g &= \xi_3 \mathbb{Z} t + \mathbb{R} g, \\
\Lambda_b &= \xi_1 \mathbb{Z} v + \mathbb{R} b, & \Lambda_e &= \xi_2 \mathbb{Z} y + \mathbb{R} e, & \Lambda_h &= \xi_3 \mathbb{Z} t + \mathbb{R} h, \\
\Lambda_c &= \xi_1 \mathbb{Z} w + \mathbb{R} c, & \Lambda_f &= \xi_2 \mathbb{Z} z + \mathbb{R} f, & \Lambda_i &= \xi_3 \mathbb{Z} t + \mathbb{R} i.
\end{aligned}$$

so that if $u_a = u$, $u_b = v$, $u_c = w$, $u_d = x$, $u_e = y$, $u_f = z$, $u_g = u_h = u_i = t$; and $\xi_a = \xi_b = \xi_c = \xi_1$, $\xi_d = \xi_e = \xi_f = \xi_2$, and $\xi_g = \xi_h = \xi_i = \xi_3$, then for each $p \in P$, $\Lambda_p = \xi_p \mathbb{Z} u_p + \mathbb{R} p$. Thus, for each $p \in P$, Λ_p is a grille with $\mathbb{R} p \subseteq \Lambda_p \subseteq \pi_p$. Also, note that if $p, p' \in P$ and $\lambda_p \cap \lambda_{p'} = \lambda_q$, then $\xi_p = \xi_{p'}$ if and only if $q \in P'$. Define

$$X = \bigcup_{p \in P} \Lambda_p \quad \text{and} \quad X_0 = \bigcup_{p \in P} \mathbb{R} p.$$

For each $p \in P$, consider X near the line $\mathbb{R} q$. We have

$$\begin{aligned}
q \notin \lambda_p &\implies \mathbb{R} q \cap \Lambda_p = \mathbb{R} q \cap \pi_p = \{0\}, \\
q \in \lambda_p \setminus \{p\} &\implies \mathbb{R} q \cap \Lambda_p = \mathbb{R} q \cap (\xi_p \mathbb{Z} u_p + \mathbb{R} p) = \xi_p \left| \frac{u_p - p}{q - p} \right| \mathbb{Z} q, \\
\text{and} \quad q = p &\implies \mathbb{R} q \cap \Lambda_p = \mathbb{R} q.
\end{aligned}$$

Now, for each $q \in P \cup P'$ define $X_q \in \mathcal{R}_c(\Gamma)$ by

$$q \in P \implies X_q = \mathbb{R}q \cup \bigcup_{\substack{p \in P\setminus\{q\} \\ q \in \lambda_p}} \Lambda_p = \mathbb{R}q \cup \bigcup_{\substack{p \in P\setminus\{q\} \\ q \in \lambda_p}} \xi_p \left|\frac{u_p - p}{q-p}\right| \mathbb{Z}q + \mathbb{R}p,$$

$$\text{and } q \in P' \implies X_q = \bigcup_{\substack{p \in P \\ q \in \lambda_p}} \Lambda_p = \bigcup_{\substack{p \in P \\ q \in \lambda_p}} \xi_p \left|\frac{u_p - p}{q-p}\right| \mathbb{Z}q + \mathbb{R}p.$$

Note that if $q \in P \cup P' \setminus \{t\}$, then $\{p \in P\setminus\{q\} : q \in \lambda_p\}$ has two elements, say p_1, p_2. Then $q = \lambda_{p_1} \cap \lambda_{p_2}$, so $\xi_{p_1} = \xi_{p_2}$ if and only if $q \in P'$. Moreover, for each $p \in P$, $|(u_p - p)/(q-p)| \in \mathbb{Q}$, so that $\xi_{p_1}|(u_{p_1} - p_1)/(q-p_1)|$ and $\xi_{p_2}|(u_{p_2} - p_2)/(q-p_2)|$ are rationally dependent if and only if $q \in P'$. Hence, if $q \in P$, then X_q is an instance of Example B.2.3, in the rationally independent case, and if $q \in P' \setminus \{t\}$, then X_q is an instance of Example B.2.2. Thus, for each $q \in P \cup P' \setminus \{t\}$, $\mathcal{I}(X_q)$ is complemented. Finally, $X_t = (\xi_3\mathbb{Z}t + \mathbb{R}g) \cup (\xi_3\mathbb{Z}t + \mathbb{R}h) \cup (\xi_3\mathbb{Z}t + \mathbb{R}\imath)$, which can be shown to have complemented kernel using [4, Theorem 3.11].

We now indicate how the localization will occur. (A formal procedure will be described in Theorem B.4.4 below.) Firstly, take a neighbourhood U of $0 \in \mathbb{R}^n$ such that $X \cap U = X_0 \cap U$. For each $q \in P \cup P'$, take a neighbourhood V_q of $\mathbb{R}q$ such that $X \cap V_q = (X_0 \cup X_q) \cap V_q$. Moreover, if we take all the neighbourhoods V_q small enough, we can assume that for distinct $q, q' \in P \cup P'$, $V_q \cap V_{q'} \subseteq U$, and then for each q, $X \cap V_q \setminus U = X_q \cap V_q \setminus U$.

B.4. Banach Space Complements to Ideals in $L^1(\mathbb{R}^n)$

We now need some results to deduce the complementedness of $\mathcal{I}(X)$ from the complementedness of the ideals $\mathcal{I}(X_q)$. Note that we are looking for a splitting map for the exact sequence

$$0 \to \mathcal{I}(X) \xrightarrow{\iota} L^1(G) \xrightarrow{Q} L^1(G)/\mathcal{I}(X) \to 0$$

where $\iota : \mathcal{I}(X) \to L^1(G)$ is the inclusion map, $Q : L^1(G) \to L^1(G)/\mathcal{I}(X)$ is the quotient mapping, and a *splitting map* is a right inverse for Q, that is, some

$T : L^1(G)/\mathcal{I}(X) \rightarrow L^1(G)$ such that $Q \circ T$ is the identity on $L^1(G)/\mathcal{I}(X)$. Also note that by Proposition 1.6.10, $L^1(G)/\mathcal{I}(X) \cong A(X)$, and so it turns out that we are looking for a continuous linear right inverse to the restriction mapping $\rho_X : A(\Gamma) \rightarrow A(X)$. Such a right inverse to ρ_X we will also refer to as a splitting map.

It can be seen that the results of Section 1.6 supplied a right inverse to ρ_X, since there we proved that $A(X) = A(\Gamma)|_X$. The present task is considerably more difficult.

The same can be said of the question as to whether $\mathcal{I}(X)/\mathcal{I}(W)$ is complemented in $L^1(G)/\mathcal{I}(W)$, for $X, W \in \mathcal{R}_c(\Gamma)$ with $X \subseteq W$. In this case, $L^1(G)/\mathcal{I}(W) \cong A(W)$, $\mathcal{I}(X)/\mathcal{I}(W) \cong \mathcal{I}_{A(W)}(X)$, and $A(W)/\mathcal{I}_{A(W)}(X) \cong A(X)$, so we are looking for a continuous linear right inverse to the restriction mapping $\rho_X : A(W) \rightarrow A(X)$.

With such concepts, we have the following.

B.4.1. Lemma. *Suppose $X, W \in \mathcal{R}_c(\Gamma)$ and $X \subseteq W$. If $\mathcal{I}(X)$ is complemented in $L^1(G)$, then $\mathcal{I}_{A(W)}(X)$ is complemented in $A(W)$.*

Proof. If $T : A(X) \rightarrow A(\Gamma)$ is a splitting map, then $\rho_W \circ T : A(X) \rightarrow A(W)$ is clearly a splitting map. ∎

B.4.2. Lemma. *Suppose $X, W \in \mathcal{R}_c(\Gamma)$, $X \subseteq W$ and $\mathcal{I}(W)$ is complemented in $L^1(G)$. If $\mathcal{I}_{A(W)}(X)$ is complemented in $A(W)$, then $\mathcal{I}(X)$ is complemented in $L^1(G)$.*

Proof. We have splitting maps $T_X : A(X) \rightarrow A(W)$ and $T_W : A(W) \rightarrow A(\Gamma)$. Then $T = T_W \circ T_X : A(X) \rightarrow A(\Gamma)$ is a right inverse of ρ_X. ∎

B.4.3. Lemma. *Suppose $W, X_1, \ldots, X_n, X \in \mathcal{R}_c(\Gamma)$ and $F_1, \ldots, F_n \in B(W)$ are such that*

(i) $X \subseteq \bigcup_1^n X_k \subseteq W$,

(ii) for each k, either $X \subseteq X_k$ or $X_k \subseteq X$,

(iii) for each k, $F_k(X \triangle X_k) = \{0\}$,

(iv) $\sum_1^n F_k(\gamma) = 1$ for $\gamma \in X$, and

(v) each $\mathcal{I}_{A(W)}(X_k)$ is complemented in $A(W)$,

then $\mathcal{I}_{A(W)}(X)$ is complemented in $A(W)$.

Proof. Let $T_k : A(X_k) \to A(\Gamma)$ $(1 \leq k \leq n)$ be splitting maps, and let $f \in A(X)$. For each k such that $X_k \subseteq X$, put $f_k = F_k \cdot T_k(f|_{X_k}) \in A(W)$, so that if $\gamma \in X$, then $f_k(\gamma) = F_k(\gamma)f(\gamma)$. For each k such that $X \subseteq X_k$, we have $F_k|_X \cdot f \in A(X)$ is zero on $X \cap \overline{X_k \setminus X}$, so we can define

$$f'_k(\gamma) = \begin{cases} F_k(\gamma)f(\gamma) & \text{if } \gamma \in X, \\ 0 & \text{if } \gamma \in X_k \setminus X. \end{cases}$$

Then f'_k is continuous, and since $\overline{X_k \setminus X} \in \mathcal{R}_c(\Gamma)$ and $f'_k|_{\overline{X_k \setminus X}} = 0 \in A(\overline{X_k \setminus X})$. It follows that $f'_k \in A(X_k)$. Moreover, providing we choose charts for X_k such that each chart has range either within X or $\overline{X_k \setminus X}$ (or both) then

$$\begin{aligned}\|f'_k\|_{A(X_k)} &= \|f'_k|_X\|_{A(X)} + \|f'_k|_{X \cap \overline{X_k \setminus X}}\|_{A(X \cap \overline{X_k \setminus X})} + \|f'_k|_{\overline{X_k \setminus X}}\|_{A(\overline{X_k \setminus X})} \\ &= \|F_k|_X \cdot f\|_{A(X)} + 0 + 0.\end{aligned}$$

Now put $f_k = T_k(f'_k) \in A(W)$, so that if $\gamma \in X$, then $f_k(\gamma) = f'_k(\gamma) = F_k(\gamma)f(\gamma)$. Now define $T(f) = \sum_1^n f_k$, then $T : A(X) \to A(W)$ is linear and $\|T(f)\| \leq \sum_1^n \|T_k\|\|F_k\|\|f\|$, so that T is continuous. Finally, for each $\gamma \in X$, $T(f)(\gamma) = \sum_1^n F_k(\gamma)f(\gamma) = f(\gamma)$, so that $T : A(X) \to A(W)$ is the desired splitting map. ∎

Now suppose $X \subseteq \mathbb{R}^n$ is a union of grilles $\Lambda_1, \ldots, \Lambda_m$. For each $1 \leq k \leq m$, let π_k be the plane containing Λ_k. Also let $\mathcal{L}$ be the set of all lines that occur as the intersection of two (or more) planes π_k, and let $\mathcal{P}$ be the set of all points that occur as the intersection of two (or more) planes π_k. For each $\lambda \in \mathcal{L}$, and each

$1 \leq k \leq m$ put

$$X'_\lambda = \bigcup\{\Lambda_k : 1 \leq k \leq m, \lambda \not\subseteq \Lambda_k, \lambda \subseteq \pi_k\}$$

$$X_\lambda = \begin{cases} \lambda \cup X'_\lambda & \text{if } \lambda \subseteq X \\ X'_\lambda & \text{if } \lambda \not\subseteq X \end{cases}$$

$$X_k = \bigcup\{\Lambda_j : \pi_j = \pi_k\}.$$

B.4.4. Theorem. *With such X, $\mathcal{I}(X)$ is complemented if and only if each of the ideals in*

$$\{\mathcal{I}(X_\lambda) : \lambda \in \mathcal{L}\} \cup \{\mathcal{I}(X_k) : 1 \leq k \leq m\}$$

is complemented.

Proof. Put $W = \bigcup_1^m \pi_k$, then $X \subseteq W \in \mathcal{R}_c(\Gamma)$ and as an instance of Example B.2.1, $\mathcal{I}(W)$ is complemented. Thus, by Lemma B.4.2, it is enough to show that $\mathcal{I}_{A(W)}(X)$ is complemented in $A(W)$ if and only if each $\mathcal{I}(X_k)$ is complemented in $A(W)$.

For each $p \in \mathcal{P}$, put $X_p = \bigcup\{\lambda \in \mathcal{L} : p \in \lambda \subseteq X\}$. This is as in Example B.2.1, so $\mathcal{I}(X_p)$ is complemented. Let $U \subseteq \mathbb{R}^n$ be an open neighbourhood of 0 such that for each $p \in \mathcal{P}$, $(p + 2U) \cap X = (p + 2U) \cap X_p$. Then in particular, for each $p_1, p_2 \in \mathcal{P}$, $(p_1 + U) \cap (p_2 + U) = \varnothing$.

Now, if $\lambda \in \mathcal{L}$ and $1 \leq k \leq m$, we consider four cases :

(i) $\lambda \cap \pi_k = \varnothing$,

(ii) $\lambda \cap \pi_k = \{p\}$, for some $p \in \mathcal{P}$,

(iii) $\lambda \subseteq \pi_k$, but $\lambda \not\subseteq \Lambda_k$, or

(iv) $\lambda \subseteq \Lambda_k$.

In the first case above, there exists $V_{\lambda,k} \subseteq \mathbb{R}^n$, a neighbourhood of 0, such that $(\lambda + V_{\lambda,k}) \cap (\pi_k + V_{\lambda,k}) = \varnothing$. In the second case, consider V_0, the closed unit ball in $\mathbb{R}^n$. Then $(\lambda + V_0) \cap (\pi_k + V_0)$ is a compact neighbourhood of p. Putting $V_{\lambda,k} = \varepsilon V_0$, then $(\lambda + V_{\lambda,k}) \cap (\pi_k + V_{\lambda,k}) \subseteq U + p$, for some sufficiently small $\varepsilon > 0$. In the third case, $\Lambda_k \subseteq X_\lambda$, put $V_{\lambda,k} = \mathbb{R}^n$. In the fourth case, there exists $V_{\lambda,k} \subseteq \mathbb{R}^n$, a neighbourhood of 0, such that $(\lambda + V_{\lambda,k}) \cap (\Lambda_k \setminus \lambda) = \varnothing$.

Now put

$$V = \bigcap_{\substack{\lambda \in \mathcal{L} \\ 1 \le k \le m}} V_{\lambda,k}$$

and if $p \in \mathcal{P}$, define

$$V_p = \bigcup_{\substack{\lambda \in \mathcal{L} \\ 1 \le k \le m \\ \lambda \cap \pi_k = \{p\}}} (\lambda + V) \cap (\pi_k + V)$$

and put $\tilde{V} = \bigcup_{p \in \mathcal{P}} (V_p - p)$. Then $\tilde{V}$ is a compact subset of U. (If $\mathcal{P} = \emptyset$, then $\tilde{V} = \emptyset$—this will not concern us, as we will only use $\tilde{V}$ if there is some $p \in \mathcal{P}$.)

Now, for each $p \in P$, let $f_p \in A(\mathbb{R}^n)$ be such that $f_p(p + \tilde{V}) = \{1\}$ and $f_p(\mathbb{R}^n \setminus (p + U)) = \{0\}$. Then $\{x \in X : f_p(x) \neq 0\} \subseteq X_p$. For each $\lambda \in \mathcal{L}$, let $F'_\lambda \in B(\mathbb{R}^n)$ be such that $F'_\lambda(\lambda) = \{1\}$ and $F'_\lambda(\mathbb{R}^n \setminus (\lambda + V)) = \{0\}$ and put $F_\lambda = \left(1 - \sum_{p \in \mathcal{P}} f_p\right) F'_\lambda \in B(\mathbb{R}^n)$. Define $g = \left(1 - \sum_{p \in \mathcal{P}} f_p\right)\left(1 - \sum_{\lambda \in \mathcal{L}} F'_\lambda\right) \in B(\mathbb{R}^n)$. Then $\sum_{p \in \mathcal{P}} f_p + \sum_{\lambda \in \mathcal{L}} F_\lambda + g = 1$.

Note that for distinct $p, q \in \mathcal{P}$, $f_p(x) \neq 0 \implies f_q(x) = 0$, so that $\sum_{p \in \mathcal{P}} f_p(x) = 1$ for all $\gamma \in \mathcal{P} + \tilde{V}$. Also, if $\lambda \in \mathcal{L}$ and $1 \le k \le m$ with $\lambda \not\subseteq \pi_k$, then we have case (i) or case (ii) above. Thus $(\lambda + V) \cap (\pi_k + V) = \emptyset$ or $(\lambda + V) \cap (\pi_k + V) \subseteq \tilde{V} + p$. In either case $(\lambda + V) \cap (\pi_k + V) \subseteq \tilde{V} + \mathcal{P}$. Hence, if $x \in \pi_k + V$ is such that $F'_\lambda(x) = 0$, then $\sum_{p \in \mathcal{P}} f_p(x) = 1$. It follows that $F_\lambda = 0$ on π_k. Moreover, if $\lambda \subseteq \Lambda_k$, then $\lambda + V$ and $\Lambda_k \setminus \lambda$ are disjoint, so that $F_\lambda = 0$ on $\Lambda_k \setminus \lambda$. Thus $\{\gamma \in X : F_\lambda(\gamma) \neq 0\} \subseteq X_\lambda$.

Also, if $\lambda, \lambda' \in \mathcal{L}$ are distinct, then there exists some $1 \le k \le m$ with $\lambda \not\subseteq \pi_k$ and $\lambda' \subseteq \pi_k$. Then $(\lambda + V) \cap (\lambda' + V) \subseteq (\lambda + V_{\lambda,k}) \cap (\pi_k + V_{\lambda,k}) \subseteq \tilde{V} + \mathcal{P}$. Hence if $x \in \bigcup \mathcal{L}$, then either $\sum_{p \in \mathcal{P}} f_p(x) = 1$ or $\sum_{\lambda \in \mathcal{L}} F_\lambda(x) = 1$, and so $g = 0$ on $\bigcup \mathcal{L}$. Hence, for each $1 \le k \le m$, we can define $g_k : W \to \mathbb{C}$ by

$$g_k(\gamma) = \begin{cases} g(\gamma) & \text{if } \gamma \in \pi_k, \\ 0 & \text{otherwise.} \end{cases}$$

Then $g_k|_{\pi_k} = g|_{\pi_k} \in B(\pi_k)$ and if $j \neq k$, then $g_k|_{\pi_j} = 0 \in B(\pi_j)$, so that $g_k \in B(W)$, and $\{x \in X : g_k(x) \neq 0\} \subseteq \pi_k \setminus (\bigcup \mathcal{L}) \subseteq X_k$. Also $\sum_1^m g_k = g$, so

$\sum_{p\in\mathcal{P}} \tilde{f}_p|_W + \sum_{\lambda\in\mathcal{L}} \tilde{F}_\lambda|_W + \sum_{1\le k\le m} g_k = 1$. We can now apply Lemma B.4.3 to conclude that $\mathcal{I}_{A(W)}(X)$ is complemented in $A(W)$, as required.

Conversely, suppose $\mathcal{I}(X)$ is complemented. Then for each $1 \le k \le m$, $\mathcal{I}_{A(\pi_k)}(X \cap \pi_k)$ is complemented in $A(\pi_k)$. However, $A(\pi_k) \cong L^1(\mathbb{R}^2)$, and so we can use the characterization of the complemented ideals in $L^1(\mathbb{R}^2)$ from [1]. In particular, if we have a hull $X' = \bigcup_{\mathbb{I}} X'_k \in \mathcal{R}_c(\mathbb{R}^2)$ such that $\mathcal{I}(X')$ is complemented, then if we put $\mathbb{I}' = \{k \in \mathbb{I} : X'_k$ is a grille$\}$, then $\mathcal{I}(\bigcup_{\mathbb{I}'} X'_k)$ is complemented in $A(\pi_k)$. Consequently, $\mathcal{I}_{A(\pi_k)}(X_k)$ is complemented in $A(\pi_k)$, and so $\mathcal{I}(X_k)$ is complemented in $L^1(\mathbb{R}^n)$.

Similarly, if $\lambda \in \Lambda$ and $\lambda \subseteq \pi_k$, then we can show that $\mathcal{I}_{A(\pi_k)}(X_\lambda \cap \pi_k)$ is complemented in $A(\pi_k)$. Also, for each $p \in \mathcal{P}$, $\mathcal{I}(X_p \cap X_\lambda)$ is complemented. Clearly

$$X_\lambda \subseteq X \cup \bigcup_{\substack{1\le k\le m\\ \lambda\subseteq\pi_k}} (X_\lambda \cap \pi_k),$$

and we can use the functions

$$g'_k = g_k + \sum_{p\in\mathcal{P}\cap\pi_k\setminus\lambda} f_p,$$

$$\text{and } g'_0 = 1 - \sum_{\substack{1\le k\le m\\ \lambda\subseteq\pi_k}} g'_k$$

from which point we can apply Lemma B.4.3. ∎

It is now a simple matter to apply this result to the situation of the hull constructed in Section B.3, providing a positive answer to [2, Question 4.1].

B.4.5. Corollary. *Let $\{\Lambda_p\}_{p\in P}$ be the subgroups as constructed in Section B.3, $X = \bigcup_{p\in P} \Lambda_p$, and for each $q \in P$, $Y_q = \bigcup_{p\in P\setminus\{q\}} \Lambda_p$. Then $\mathcal{I}(X)$ is complemented, but for each $q \in P$, $\mathcal{I}(Y_q)$ is not complemented.* ∎

It should be noted that despite this, it is possible to use the methods of [4] to build a chain of projections which show the ideal $\mathcal{I}(X)$ to be complemented. For this, let $p_1, \ldots, p_9$ be the elements of P, and for $1 \le k \le 9$ let $X_k = \mathbb{R}p_k$ and

$X_{k+9} = \Lambda_{p_k} \setminus X_k$, so that $X = \bigcup_1^{18} X_k$. For $0 \le k \le 9$, put $W_k = \bigcup_1^{9+k} X_j$. Each of $X_1, \ldots, X_9$ is a Euclidean coset, and so as in Example B.2.1, $\mathcal{I}(W_0)$ is complemented. In fact, there exists a chain of projections $L^1(G) \to \mathcal{I}(X_1) \to \cdots \to \mathcal{I}(W_0)$. To complete the chain of projections, we can use a method equivalent to that described in the introduction to Section 4 of [4]. For $1 \le k \le 9$, let $\pi_k = \operatorname{span} \Lambda_{p_k}$, the plane containing the grille Λ_{p_k} and put $\widetilde{W}_k = W_0 \cup \bigcup_1^k \pi_j$. Then for each k, $\mathcal{I}(\widetilde{W}_k)$ is complemented in $L^1(G)$ and $\mathcal{I}_{A(\pi_k)}(\pi_k \cap X)$ is complemented in $A(\pi_k)$. Thus if $\mathcal{I}(W_{k-1})$ is complemented, there exists a splitting map $A(W_{k-1}) \to L^1(G)$, and hence a splitting map $A(W_{k-1}) \to A(\widetilde{W}_{k-1})$. Now, $W_{k-1} \cup (\pi_k \cap X) = W_k$ and $\pi_k \cap \widetilde{W}_{k-1} \subseteq W_0 \subseteq W_k$, and so we can combine this previous splitting map with a splitting map $A(\pi_k \cap X) \to A(\pi_k)$ to give a splitting map $A(W_k) \to A(\widetilde{W}_k)$. Composing this with a splitting map $A(\widetilde{W}_k) \to L^1(G)$ yields a splitting map $A(W_k) \to L^1(G)$, which gives the projection required.

B.5. Building a Hull in $\mathbb{R}^4$

Whereas the previous example had the behaviour of Example B.2.3 on each specified line, we now specify a construction where we have the behaviour of Example B.2.4 on each specified line.

Due to this, we need to consider having an affine image of $(\mathbb{R} \times \mathbb{Z}) \cup (\mathbb{Z} \times \mathbb{R})$ on each plane, instead of just $\mathbb{R} \times \mathbb{Z}$. Such a set we will refer to as a *grid*, which will generally be of the form $\big((\mathbb{R}a + \mathbb{Z}b) \cup (\mathbb{Z}a + \mathbb{R}b)\big) + c$, for some $a, b, c \in \mathbb{R}^n$ with a and b linearly independent. Moreover, this needs to be done so that at each line where the three planes intersect we have the situation of Example B.2.4, with one of these planes taking the rôle of the grid plane and the other two planes behaving as the grille planes. For this reason, we use $\widetilde{\mathbb{Z}}$ in place of $\mathbb{Z}$, so that if a plane π intersects $\mathbb{R}a + \mathbb{R}b$ in $\mathbb{R}a$, then $\big((\mathbb{R}a + \widetilde{\mathbb{Z}}b) \cup (\widetilde{\mathbb{Z}}a + \mathbb{R}b)\big) \cap \pi = \widetilde{\mathbb{Z}}a$, so that with respect to its intersection with π, $(\mathbb{R}a + \widetilde{\mathbb{Z}}b) \cup (\widetilde{\mathbb{Z}}a + \mathbb{R}b)$ behaves like the grille $\widetilde{\mathbb{Z}}a + \mathbb{R}b$.

If we try using the same arrangement of planes as in Section B.3, it is difficult to control the situation along the lines $\mathbb{R}t, \ldots, \mathbb{R}z$. It may be possible to ignore this initially, and then to add extra cosets to "cover up" the non-uniformly separated bits, in the way that the uncomplementary behaviour of Example B.2.2 can be covered up either by Λ_3 or $\Lambda_4 \cup \Lambda_5$, in Examples B.2.3 and B.2.4 respectively. Rather than attempt this, we will resort to a different "tree-planting" that avoids such points as $t, \ldots, z$. This is done in $\mathbb{R}^3$, instead of $\mathbb{R}^2$, so that the resulting hull will be a subset of $\mathbb{R}^4$.

Define points $P = \{a, \ldots, j\} \in \mathbb{R}^3 \times \{1\} \subseteq \mathbb{R}^4$ as follows,

$$\begin{aligned}
a &= (-1, 1, 1, 1) & f &= (2, -1, 2, 1) \\
b &= (1, 1, -1, 1) & g &= (-1, 0, 0, 1) \\
c &= (2, 1, 2, 1) & h &= (1/2, 1/2, -1, 1) \\
d &= (-1, -1, -1, 1) & i &= (1, 0, 0, 1) \\
e &= (1, -1, 1, 1) & j &= (1/2, -1/2, 1, 1)
\end{aligned}$$

(Here $abde$ is a regular tetrahedron, c and f are found by extending $\overline{ab}$ and $\overline{de}$, respectively, by half their lengths. The points g and i are midpoints of the sides ad and be, Then c, f, g, i are coplanar, and h and j are the intersections of this plane with $\overline{bd}$ and $\overline{ae}$, respectively.) We have ten lines :

$$\begin{aligned}
\lambda_a &= \overline{abc} & \lambda_d &= \overline{dga} & \lambda_g &= \overline{gjf} & \lambda_i &= \overline{bie} \\
\lambda_b &= \overline{bhd} & \lambda_e &= \overline{eja} & \lambda_h &= \overline{hif} & \lambda_j &= \overline{jic} \\
\lambda_c &= \overline{chg} & \lambda_f &= \overline{fed}
\end{aligned}$$

which do not intersect anywhere except for on P. Moreover, we again have that $p \in P$ is in three lines λ_q, and each line λ_p contains three points $q \in P$. We desire a grid on each plane $\pi_p = \operatorname{span} \lambda_p$. We next consider a procedure that allows us to construct all twenty grilles so that each $\mathbb{R}p$ intersects four grilles $\Xi_1, \ldots, \Xi_4$ so that $\Xi_1 \cup \cdots \cup \Xi_4$ is an instance of Example B.2.4.

Consider the cycles

$$a \to b \to d \to a$$
$$b \to h \to i \to b$$
$$c \to g \to j \to c$$
$$d \to g \to f \to d$$
$$e \to j \to i \to e$$
$$a \to c \to h \to f \to e \to a$$

These have the property that if $x \to y$ and $x \to z$ both occur in the cycles, then $\lambda_x = \overline{xyz}$. Conversely, if $\lambda_x = \overline{xyz}$, for $x, y, z \in P$, then we have both $x \to y$ and $x \to z$ in the cycles.

Now, let $\xi_1 \in \mathbb{R}$ and put $\Xi_a^1 = \xi_1 \widetilde{\mathbb{Z}} a + \mathbb{R}c$, then $|(a-c)/(b-c)| = 3$, so by the note at the start of Section B.3, $\Xi_a^1 = 3\xi_1 \widetilde{\mathbb{Z}} b + \mathbb{R}c$. Put $\Xi_b^1 = 3\xi_1 \widetilde{\mathbb{Z}} b + \mathbb{R}h$. Then $|(b-h)/(d-h)| = \frac{1}{3}$, so $\Xi_b^1 = \xi_1 \widetilde{\mathbb{Z}} d + \mathbb{R}g$. Put $\Xi_d^1 = \xi_1 \widetilde{\mathbb{Z}} d + \mathbb{R}g$, then $\Xi_d^1 = |(d-g)/(a-g)|\xi_1 \widetilde{\mathbb{Z}} a + \mathbb{R}g = \xi_1 \widetilde{\mathbb{Z}} a + \mathbb{R}g$. This intersects $\widetilde{\mathbb{Z}} a$ in the same coset that Ξ_a^1 does. Similarly if $\xi_2 \in \mathbb{R}$, and $\Xi_b^2 = \xi_1 \widetilde{\mathbb{Z}} b + \mathbb{R}d$, then following around the cycle yields eventually $\Xi_i^2 = \xi_2 \widetilde{\mathbb{Z}} b + \mathbb{R}e$. This procedure is successful for every cycle given above. Summarizing the necessary arithmetic, we have:

$$\left|\frac{a-c}{b-c}\right| \cdot \left|\frac{b-h}{d-h}\right| \cdot \left|\frac{d-g}{a-g}\right| = 3 \cdot \frac{1}{3} \cdot 1 = 1,$$
$$\left|\frac{b-d}{h-d}\right| \cdot \left|\frac{h-f}{i-f}\right| \cdot \left|\frac{i-e}{b-e}\right| = \frac{4}{3} \cdot \frac{3}{2} \cdot \frac{1}{2} = 1,$$
$$\left|\frac{c-h}{g-h}\right| \cdot \left|\frac{g-f}{j-f}\right| \cdot \left|\frac{j-i}{c-i}\right| = 1 \cdot 2 \cdot \frac{1}{2} = 1,$$
$$\left|\frac{d-a}{g-a}\right| \cdot \left|\frac{g-j}{f-j}\right| \cdot \left|\frac{f-e}{d-e}\right| = 2 \cdot 1 \cdot \frac{1}{2} = 1,$$
$$\left|\frac{e-a}{j-a}\right| \cdot \left|\frac{j-c}{i-c}\right| \cdot \left|\frac{i-b}{e-b}\right| = \frac{4}{3} \cdot \frac{3}{2} \cdot \frac{1}{2} = 1,$$
and
$$\left|\frac{a-b}{c-b}\right| \cdot \left|\frac{c-g}{h-g}\right| \cdot \left|\frac{h-i}{f-i}\right| \cdot \left|\frac{f-d}{e-d}\right| \cdot \left|\frac{e-j}{a-j}\right| = 2 \cdot 2 \cdot \frac{1}{2} \cdot \frac{3}{2} \cdot \frac{1}{3} = 1.$$

Note. I do not know how coincidental this situation is—certainly we can shift our points around somewhat with no significant effect. I do not know whether an alternative set of assignments of the ten lines to the ten points would work or whether an alternative set of cycles would work. This is the first configuration I tried, and it worked.

We now specify $\xi_1, \ldots, \xi_6 \in \mathbb{R}$ to be pairwise rationally independent, so that on each of the lines $\mathbb{R}a, \ldots, \mathbb{R}j$, the intersection with the cosets Ξ^*_* occur in pairs, at rationally independent spacings.

This allows us to explicitly specify all twenty grilles.

$$
\begin{aligned}
\Xi^1_a &= \xi_1\widetilde{\mathbb{Z}}a + \mathbb{R}c = 3\xi_1\widetilde{\mathbb{Z}}b + \mathbb{R}c & \Xi^4_d &= \xi_4\widetilde{\mathbb{Z}}d + \mathbb{R}a = 2\xi_4\widetilde{\mathbb{Z}}g + \mathbb{R}a\\
\Xi^1_b &= 3\xi_1\widetilde{\mathbb{Z}}b + \mathbb{R}h = \xi_1\widetilde{\mathbb{Z}}d + \mathbb{R}h & \Xi^4_g &= 2\xi_4\widetilde{\mathbb{Z}}g + \mathbb{R}j = 2\xi_4\widetilde{\mathbb{Z}}f + \mathbb{R}j\\
\Xi^1_d &= \xi_1\widetilde{\mathbb{Z}}d + \mathbb{R}g = \xi_1\widetilde{\mathbb{Z}}a + \mathbb{R}g & \Xi^4_f &= 2\xi_4\widetilde{\mathbb{Z}}f + \mathbb{R}e = \xi_4\widetilde{\mathbb{Z}}d + \mathbb{R}e\\
\Xi^2_b &= \xi_2\widetilde{\mathbb{Z}}b + \mathbb{R}d = 4/3\xi_2\widetilde{\mathbb{Z}}h + \mathbb{R}d & \Xi^5_e &= \xi_5\widetilde{\mathbb{Z}}e + \mathbb{R}a = 4/3\xi_5\widetilde{\mathbb{Z}}j + \mathbb{R}a\\
\Xi^2_h &= 4/3\xi_2\widetilde{\mathbb{Z}}h + \mathbb{R}f = 2\xi_2\widetilde{\mathbb{Z}}i + \mathbb{R}f & \Xi^5_j &= 4/3\xi_5\widetilde{\mathbb{Z}}j + \mathbb{R}c = 2\xi_5\widetilde{\mathbb{Z}}i + \mathbb{R}c\\
\Xi^2_i &= 2\xi_2\widetilde{\mathbb{Z}}i + \mathbb{R}e = \xi_2\widetilde{\mathbb{Z}}b + \mathbb{R}e & \Xi^5_i &= 2\xi_5\widetilde{\mathbb{Z}}i + \mathbb{R}b = \xi_5\widetilde{\mathbb{Z}}e + \mathbb{R}b\\
\Xi^3_c &= \xi_3\widetilde{\mathbb{Z}}c + \mathbb{R}h = \xi_3\widetilde{\mathbb{Z}}g + \mathbb{R}h & \Xi^6_a &= \xi_6\widetilde{\mathbb{Z}}a + \mathbb{R}b = 2\xi_6\widetilde{\mathbb{Z}}c + \mathbb{R}b\\
\Xi^3_g &= \xi_3\widetilde{\mathbb{Z}}g + \mathbb{R}f = 2\xi_3\widetilde{\mathbb{Z}}j + \mathbb{R}f & \Xi^6_c &= 2\xi_6\widetilde{\mathbb{Z}}c + \mathbb{R}g = 4\xi_6\widetilde{\mathbb{Z}}h + \mathbb{R}g\\
\Xi^3_j &= 2\xi_3\widetilde{\mathbb{Z}}j + \mathbb{R}i = \xi_3\widetilde{\mathbb{Z}}c + \mathbb{R}i & \Xi^6_h &= 4\xi_6\widetilde{\mathbb{Z}}h + \mathbb{R}i = 2\xi_6\widetilde{\mathbb{Z}}f + \mathbb{R}i\\
& & \Xi^6_f &= 2\xi_6\widetilde{\mathbb{Z}}f + \mathbb{R}d = 3\xi_6\widetilde{\mathbb{Z}}e + \mathbb{R}d\\
& & \Xi^6_e &= 3\xi_6\widetilde{\mathbb{Z}}e + \mathbb{R}j = \xi_6\widetilde{\mathbb{Z}}a + \mathbb{R}j.
\end{aligned}
$$

Put $X = \Xi^1_a \cup \cdots \cup \Xi^6_e$, the union of the twenty cosets above. For some neighbourhood U of $\mathbb{R}a$ we have $X \cap U = X_a \cap U$, where $X_a = \Xi^1_a \cup \Xi^6_a \cup \Xi^1_d \cup \Xi^6_e$. Here $\Xi^1_a \cup \Xi^6_a$ lies in the plane π_a, Ξ^1_d lies in the plane π_d, and Ξ^6_e lies in the plane π_e. Moreover, $\Xi^1_a \cap \Xi^1_b = \xi_1\widetilde{\mathbb{Z}}a$ and $\Xi^6_a \cap \Xi^6_c = \xi_6\widetilde{\mathbb{Z}}a$, so we have the situation of Example B.2.4. It can be similarly shown that we have such a situation near each of the lines $\mathbb{R}b, \ldots, \mathbb{R}j$, and so we can use Theorem B.4.4 to show that $\mathcal{I}(X)$ is complemented.

We now show that it is not possible to apply the methods of [4] to *any* decomposition of X. This could be done by using Theorem B.4.4, but it is easier in this case to proceed directly. Suppose we have $X = \bigcup_1^n S_k$, where each S_k belongs to the coset ring of one of the twenty cosets above, and that we have projections

$$L^1(G) \to \mathcal{I}(S_1) \to \mathcal{I}(S_1 \cup S_2) \to \cdots \to \mathcal{I}\Big(\bigcup_1^{n-1} S_k\Big) \to \mathcal{I}(X).$$

Clearly we can assume that if Ξ is one of the twenty cosets, then for distinct j, k such that $S_j, S_k \in \mathcal{R}(\Xi)$, we have that S_j and S_k are disjoint. Hence *all* of $S_1, \ldots, S_n$ are triple-wise disjoint. (That is, the intersection of any triple is empty.) Let $m \leq n$ be such that $S_{m-1}, \ldots, S_n$ each consists of a finite number of lines, whilst S_m consists of an infinite number of lines. Then $\mathcal{I}(S_m)$, $\mathcal{I}(\bigcup_1^{m-1} S_k)$, and $\mathcal{I}(\bigcup_1^{m} S_k)$ are complemented, so by [4, Proposition 1.9], $\mathcal{I}(S_m \cap \bigcup_1^{m-1} S_k)$ is complemented. However, $S_m \cap \bigcup_1^{m-1} S_k = \bigcup_1^{m-1}(S_m \cap S_k)$ is discrete and the sets $S_m \cap S_k$ $(1 \leq k < m)$ are disjoint, so by [4, Theorem 2.3], these sets are uniformly separated. It is now a simple, but tedious, task to show that this is not possible, and so there is no such chain of projections.

whether an alternative set of cycles would work. This is the first configuration I tried, and it worked.

We now specify $\xi_1, \ldots, \xi_6 \in \mathbb{R}$ to be pairwise rationally independent, so that [illegible]

This allows us to explicitly specify all twenty groups.

[illegible]

$$\Xi^5_5 = \xi_3\bar{\mathbb{Z}}c + \mathbb{R}h = \xi_5\bar{\mathbb{Z}}g + \mathbb{R}h \qquad \Xi^5_1 = \xi_4\bar{\mathbb{Z}}a + \mathbb{R}b = 2\xi_5\bar{\mathbb{Z}}c + \mathbb{R}b$$

$$\Xi^5_6 = \xi_5\bar{\mathbb{Z}}g + \mathbb{R}f = 2\xi_5\bar{\mathbb{Z}}j + \mathbb{R}f \qquad \Xi^5_2 = 3\xi_5\bar{\mathbb{Z}}c + \mathbb{R}g = 4\xi_5\bar{\mathbb{Z}}h + \mathbb{R}g$$

$$\Xi^5_7 = 2\xi_5\bar{\mathbb{Z}}j + \mathbb{R}i = \xi_5\bar{\mathbb{Z}}e + \mathbb{R}i \qquad \Xi^5_3 = 4\xi_5\bar{\mathbb{Z}}h + \mathbb{R}i = 2\xi_5\bar{\mathbb{Z}}f + \mathbb{R}i$$

$$\Xi^5_4 = 2\xi_5\bar{\mathbb{Z}}f + \mathbb{R}d = 3\xi_5\bar{\mathbb{Z}}e + \mathbb{R}d$$

$$\Xi^6_5 = 2\xi_6\bar{\mathbb{Z}}e + \mathbb{R}j = \xi_6\bar{\mathbb{Z}}a + \mathbb{R}j.$$

Put $X = \Xi^1_a \cup \cdots \cup \Xi^6_5$, the union of the twenty cosets above. For some neighborhood U of $\mathbb{R}a$ we have $X \cap U = X_a \cap U$, where $X_a = \Xi^1_a \cup \Xi^4_a \cup \Xi^1_b \cup \Xi^6_5$. Here $\Xi^1_a \cup \Xi^6_5$ lies in the plane π_a, Ξ^1_b lies in the plane π_b, and Ξ^4_a lies in the plane π_4. Moreover, $\Xi^1_a \cap \Xi^1_b = \xi_1\bar{\mathbb{Z}}a$ and $\Xi^6_5 \cap \Xi^4_a = \xi_6\bar{\mathbb{Z}}a$, so we have the situation of Example B.2.4. It can be similarly shown that we have such a situation near each of the lines $\mathbb{R}b, \ldots, \mathbb{R}j$, and so we can use Theorem B.4.4 to show that $\mathcal{I}(X)$ is complemented.

We now show that it is not possible to apply the methods of [4] to any decomposition of X. This could be done by using Theorem B.4.4, but it is easier in this case to proceed directly. Suppose we have $X = \bigcup_1^n S_k$, where each S_k belongs to

References

[1] Dale E. Alspach, A characterization of the complemented translation-invariant subspaces of $L_1(\mathbb{R}^2)$, *J. London Math Soc. (2)* **31** (1985), 115-124.

[2] Dale E. Alspach, Complemented translation-invariant subspaces., *Springer Lecture Notes in Mathematics* **1332** (1988), 112-125.

[3] Dale E. Alspach and Alec Matheson, Projections onto translation-invariant subspaces of $L^1(\mathbb{R})$, *Trans. Amer. Math. Soc.* **277** (2) (1983), 815-823.

[4] D.E. Alspach, A. Matheson and J.M. Rosenblatt, Projections onto translation-invariant subspaces of $L^1(G)$, *J. Funct. Anal.* **59** (1984), 254-292.

[5] D.E. Alspach, A. Matheson and J.M. Rosenblatt, Separating sets by Fourier-Stieltjes transforms, *J. Funct. Anal.* **84** (1989), 297-311.

[6] W.G. Bade & P.C. Curtis, Jr, Embedding theorems for commutative Banach algebras, *Pac. J. Math.* **18** (1966), 391-409.

[7] N. Bourbaki, "Éléments de Mathématique," Livre III, "Topologie Générale," 2me ed., Hermann, Paris, 1951.

[8] F.F. Bonsall and J. Duncan, "Complete Normed Algebras," Springer-Verlag, Berlin, 1973.

[9] J.T. Burnham, Closed ideals in subalgebras of Banach algebas. I, *Proc. Amer. Math. Soc.* **32** (1972), 551-555.

[10] P.J. Cohen, On a conjecture of Littlewood and idempotent measures, *Amer. J. Math.* **82** (1960), 191-212.

[11] P.J. Cohen, On homomorphisms of group algebras, *Amer. J. Math.* **82** (1960), 213-226.

[12] J. Cuntz, Simple C*-algebras generated by isometries, *Commun. Math. Phys.* **57** (1977), 173-185.

[13] P.C. CURTIS, JR & R.J. LOY, The structure of amenable Banach algebras, *J. London Math. Soc. (2)* **40** (1989), 89-104.

[14] BRIAN FORREST, Amenability and bounded approximate identities in ideals of $A(G)$, *Ill. J. Math.* **34** (1990), 1-25.

[15] JOHN E. GILBERT, On a strong form of spectral synthesis, *Ark. Mat. (7)* **43** (1969), 571-575.

[16] LEONARD GILLMAN AND MELVIN HENRIKSON, Rings of continuous functions in which every finitely generated ideal is principal, *Trans. Amer. Math. Soc* **82** (1956), 366-391.

[17] F. GREENLEAF, "Invariant Means on Topological Groups," Van Nostrand, New York, 1969.

[18] NIELS GRØNBÆK, Amenability of weighted convolution algebras on locally compact groups, *Trans. Amer. Math. Soc. 2* **319** (2) (1990), 765-775.

[19] R.C. GUNNING AND H. ROSSI, "Analytic Functions of Several Complex Variables," Prentice-Hall, N.J., 1965.

[20] E. HEWITT AND K.A. ROSS, "Abstract Harmonic Analysis," Vol. 1, Springer-Verlag, Berlin, 1963.

[21] G. HOCHSCHILD, On the cohomology groups of an associative algebra, *Ann. Math.* **46** (1945), 58-67.

[22] B. HOST, Le Théorème des idempotents dans $B(G)$, *Bull. Soc. Math. France.* **114** (1986), 215-223.

[23] B.E. JOHNSON, Cohomology in Banach algebras, *Mem. Amer. Math. Soc.* **127** (1972).

[24] B.E. JOHNSON, Approximate diagonals and cohomology of certain annihilator Banach algebras, *Amer. J. Math.* **94** (1972), 685-698.

[25] N.J. KALTON AND G.V. WOOD, Homomorphisms of group algebras with norm less than $\sqrt{2}$, *Pac. J. Math.* **62** (1976), 439-460.

[26] L.G. KOVÁCS, On a paper of Ladislav Procházka, Чехословацкий математический журнал, т. 13 *(Czech. Math. J.)* **88** (1963), 612-618.

[27] MICHAEL J. LIDDELL, Separable Topological Algebras, I, *Trans. Amer. Math. Soc.* **195** (1974), 31-59.

[28] T.-S. LIU, A. VAN ROOIJ, J.-K. WANG, Projections and approximate identities for ideals in group algebras, *Trans. Amer. Math. Soc.* **175** (1973), 469-482.

[29] RICHARD J. LOY, Subalgebras of amenable algebras, *Proc. Centre for Math. Anal., Australian National University* **21** (1989), 288-296.

[30] A.L.T. PATERSON, "Amenability," Math. Surveys and Monographs, no. 29, American Math. Society, 1988.

[31] C.E. RICKART, "General Theory of Banach Algebras," Van Nostrand, Princeton, 1960.

[32] H. REITER, "Classical Harmonic Analysis and Locally Compact Groups," Oxford University Press, Oxford, 1968.

[33] J.-P. PIER, "Amenable Locally Compact Groups," John Wiley & Sons, New York, 1984.

[34] J.-P. PIER, "Amenable Banach Algebras," Longman, Harlow, U.K., 1988.

[35] J. ROSENBERG, Amenability of crossed products of C*-algebras, *Commun. Math. Phys.* **57** (1977), 187-191.

[36] H.P. ROSENTHAL, Projections onto translation-invariant subspaces of $L^p(G)$, *Mem. Amer. Math. Soc.* **63** (1966).

[37] W. RUDIN, "Fourier Analysis on Groups," Interscience, New York, 1962.

[38] B.M. SCHREIBER, On the coset ring and strong Ditkin sets, *Pac. J. Math.* **32** (1970), 805-812.

[39] MAHATHEVA SKANTHARAJAH, PhD Thesis, U. of Alberta.

[40] M.C. WHITE, Characters on weighted amenable groups, *Bull. Lond. Math. Soc.* **23** (1991), 375-380.

[41] G. WILLIS, The continuity of derivations and module homomorphisms, *J. Aust. Math. Soc.* **40** (1986), 299-320.